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Preference Intensity Representation and Revelation

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Abstract

This paper introduces *preference intensity functions* –an extension of (neo)classical cardinal utility functions– and characterizes by means of three simple standard axioms the class of *basic* preference intensity orderings over a finite set of general alternatives that can be represented numerically by such a function in an essentially ordinal way. Unlike utility-difference representations on finite sets, the one proposed here imposes neither behaviourally uninterpretable nor precision-demanding axioms on the preference intensity relation, while its novel uniqueness properties are pinned down in a simple way. The observable implications of this model are then analyzed. Considering general datasets that comprise (i) menus of feasible alternatives, (ii) the alternatives chosen at these menus, and (iii) the amounts of a measurable resource (e.g. money, time) that the individual has foregone in order to make these choices, it is first shown that two new testable consistency requirements on such datasets are necessary and sufficient for the latter to be *preference-intensity rationalizable.* In addition to encompassing standard rationalizability, this notion disciplines the directions that the observed differences in foregone resources can take, and at the same time allows for the decision maker's resource allocation on the same alternative to potentially vary with the menu where it was chosen. The novel concept of cardinal-utility rationalizability emerges as the special case where such resource allocation is menu-invariant.

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1 Introduction

A decision maker is modelled in this paper as being able to make preference intensity comparisons such as *"I prefer (studying or working at) university A to B more than I prefer C to D"* without assuming that he is also able to quantify these statements, e.g. by saying that he prefers A to B *twice as much* as he does C to D. The concept of a preference intensity relation is therefore approached in a purely ordinal and more general way than can be afforded by the framework of [(neo)classical] cardinal utility representations. In addition, the paper's thesis is that decision makers who are indeed capable of making such general preference intensity comparisons in a suitably consistent manner can reveal these rankings through their observable behaviour. It is argued, in particular, that existing revealed-preference methods can be extended in novel ways to provide a framework that dictates the general nature of datasets and the testable restrictions on these datasets that are both necessary and sufficient for a general preference intensity ordering over the relevant choice alternatives to be recovered.

The first part of the paper introduces the concept of a *preference intensity function* to represent numerically a decision maker's preference intensity comparisons, as captured by a binary relation that is defined on the set of *pairs* of alternatives (also known as a *quaternary* relation on the set of alternatives). Its first property, *order-preservation*, is that some alternative a is weakly preferred to b at least as much as c is to d if and only if the function assigns a weakly higher value at the pair (a, b) than at (c, d). It therefore acts on pairs of alternatives in much the same way that an ordinal utility function acts on alternatives. This function is also required to be *skew-symmetric*, i.e. to assign the same absolute value –but with opposite signs– to any two pairs that contain the same two alternatives. An appealing feature of skew-symmetry in the present context is that the function has a positive, negative or neutral sign at some pair (a, b) if and only if, respectively, a is strictly preferred to b, b is strictly preferred to a or a and b are equally good. The final property, *internal consistency*, imposes an intuitive condition on the relationship between preferences and preference intensities. In particular, it suggests that preference intensity between two alternatives is increasing as the first one is kept fixed and the second one is allowed to go down the agent's preference ranking.

Theorem 1 shows that a preference intensity relation on a finite set is representable by a preference intensity function if and only if it is a weak order that also satisfies the reversal and separability axioms. These are standard in the relevant literature, impose concrete behavioural restrictions, and are collectively weaker than the axioms involved in the characterization of utility-difference representations on finite sets (Scott, 1964). A preference intensity ordering that belongs to this general class will be referred to as *basic*. Importantly, Theorem 1 further establishes the generally ordinal nature of preference intensity functions by showing that such representations are unique up to a transformation by means of a strictly increasing function f that is also *odd*, i.e. such that $f(-z) = -f(z)$. The latter property is necessary for the skew-symmetry of preference intensity functions to be preserved, and slightly narrows the transformations afforded by this model, without, however, changing its essentially ordinal nature. Therefore, unlike utility-difference representations on finite sets that have ill-defined uniqueness properties, this model's ordinal nature is pinned down in a precise way by means of a novel, simple and intuitive subclass of general strictly increasing transformations, which, in particular, can be thought of as the ordinal counterpart of positive *linear* transformations (i.e. those that are defined only in terms of multiplication by a positive constant). Corollary 2 then identifies the precise relationship between preference-intensity and utility-difference representations by clarifying that the latter is possible if and only if the underlying relation is representable by a preference intensity function s that is *triangularly additive* in the sense that $s(a, b) = s(a, c) + s(c, b)$ for any three alternatives a, b and c.

The second part of the paper studies the testable implications of basic preference intensity relations. It is well-known that standard revealed-preference data that comprise collections of menus and the choices made at these menus do not suffice for recovering a decision maker's preference intensity ranking over the underlying alternatives. However, arguments have long been made about the possibility of making such inferences, essentially by suitably enriching the data that is made available to an analyst. For example, Suppes and Winet (1955) argued in favour of using as a proxy for measuring the intensity of preference between two alternatives variables such as the amount of money one is willing to pay (similarly, how much work one is prepared to do or how much distance one is willing to travel) in order to change one alternative for another (see also Baccelli and Mongin, 2016). More recently, response times have been used for the same purpose in behavioural and neuroeconomics, and also in revealed-preference theory, with a shorter time for choosing a over b than b over c interpreted as an indication of a higher preference intensity for a over b than for b over c (Krajbich, Oud, and Fehr, 2014; Echenique and Saito, 2017; Konovalov and Krajbich, 2017).

Building on these ideas, the proposed approach towards identifying testable implications of basic preference intensity relations builds on the revealed-preference tradition and extends it by requiring that, in addition to containing a collection of menus and choices at these menus, the *behavioural dataset* also include information about an observable measurable *resource* such as money or time that the decision maker has foregone in order to make these choices. Assuming the availability of such a dataset, the hereby proposed notion of it being *preference-intensity rationalizable* requires that, in addition to every observed chosen alternative being the most preferred feasible one, the sign in the difference between the resources allocated to alternatives a and b vs. c and d never change for any such quadruple. For example, a is revealed preferred to b at least as much as c is to d if and only if the difference between the monetary amounts allocated to a and b is always weakly higher than the difference between the monetary amounts allocated to c and d. Analogously, a is revealed preferred to b more than c is to d if the decision time difference between a and b is always *smaller* than that between c and d. Importantly, this notion of rationalizability allows the resources allocated to any one alternative to vary across the menus where that same alternative was chosen. It therefore disciplines the directions that observed differences in resources can take while allowing for the possibility that the decision maker's resource allocation is context-dependent in the above sense. Intuitively, context dependence may be relevant when the measurable resource is the decision maker's response times, even after possibly confounding factors such as the number of feasible alternatives have been controlled for, e.g. by keeping that number fixed.

Theorem 2 establishes that such a behavioural dataset is rationalizable by a preference intensity function if and only if it satisfies two new testable axioms, labelled *Cross-Modal Consistency* (CMC) and *Resource-Difference Comparability* (RDC). The former requires that the decision maker's choices between any two alternatives are consistent with his observed resource allocations. Reflecting the key new idea in the above notion of rationalizability, the second axiom requires that, for any four choice alternatives where the first is choosable in the presence of the second and the third in the presence of the fourth, if the resource difference is ever higher in the first pair than in the second, then it is always higher. A revealed-preference foundation for cardinal-utility rationalizable datasets finally emerges as a special case (Corollary 3) of this result where, in addition to satisfying the above two conditions, the decision maker's behaviour is also such that exactly the same resource value is always allocated to a given alternative, regardless of the menu where it is chosen.

The paper is organized as follows. After defining basic preference intensity relations, Section 2 provides a critical overview of cardinal utility/difference representations that motivates the introduction of preference intensity function which then follows. Section 3 turns to the revealed-preference analysis of basic preference intensity relations and, after introducing resource-augmented behavioural datasets and their rationalizability by a preference intensity function, it lays out the two axioms that characterize this notion of rationalizability. Section 4 places the paper in the literature. Appendix A illustrates Theorems 1, 2 and Corollaries 2, 3 with examples that elucidate the constructive proofs, which in turn are provided in Appendix B.

2 Representation

2.1 Preliminaries

Assumed is a set X of general choice alternatives and a reflexive binary relation $\succsim$ on the product set $X \times X$ of ordered pairs of elements of X (such an $\succsim$ is also known as a *quaternary relation* on X). While X will be assumed finite below, the concepts defined in this section do not require this assumption. The primitive intended interpretation of the statement $(a, b) \succsim (c, d)$ for alternatives $a, b, c, d \in X$ is that if the decision maker was to imagine that he had to decide between moving away from b towards a and moving from d towards c (imagining also that he was endowed with b and d), then he would weakly prefer the former transition. The relation $\succsim$ will therefore be thought of as containing information about the agent's preference intensity comparisons. Accordingly, the statement $(a, b) \succsim (c, d)$ will also be interpreted as suggesting that a is weakly better than b at least as much as c is weakly better than d (or that a is weakly worse than b at most as much as c is weakly worse than d, etc.). For such interpretations to be legitimate, however, $\succsim$ will be endowed with some additional structure.

Definition 1
A binary relation $\succsim$ on $X \times X$ is a ***basic preference intensity relation on*** *X if it satisfies:*
1. ***Weak Order****: $\succsim$ is complete and transitive;*
2. ***Reversal****: $(a, b) \succsim (c, d)$ implies $(d, c) \succsim (b, a)$;*
3. ***Separability****: $(a, c) \succsim (b, c)$ implies $(a, d) \succsim (b, d)$.*

Before discussing its three constituent properties, let us note that a basic preference intensity relation $\succsim$ on X induces a *simple* preference relation $\gtrapprox$ on that set when $\gtrapprox$ is defined by

$$a \gtrapprox b \iff (a,b) \succsim (b,a). \tag{1}$$

Since $(a,b) \succsim (b,a)$ suggests that moving from b to a is weakly preferred to moving from a to b, it is indeed natural to interpret a as weakly preferred to b if and only if this is the case. The asymmetric and symmetric parts of the preference relation $\gtrapprox$ will be denoted $\gg$ and $\approx$, respectively. Claim 4 in Appendix B shows that if $\gtrapprox$ is induced by a basic preference intensity relation in this way, then $\gtrapprox$ is itself a weak order on X. Clearly, the same weak order $\gtrapprox$ on X can be induced by several distinct basic preference intensity relations $\succsim$ on that set.

Going back to the three defining properties of a basic preference intensity relation, Weak Order and Reversal together ask that the decision maker be able to make intensity comparisons universally and consistently over all pairs of alternatives. It is worth noting, however, that although Completeness and Transitivity ensure that $\succsim$ is a complete preorder on the set of ordered pairs in $X \times X$, it is only through the additional Reversal axiom that $\succsim$ can afford the interpretation of a preference intensity weak order. Indeed, suppose that, contrary to Reversal's requirements, $(a,b) \succsim (c,d)$ and $(b,a) \succ (d,c)$ is true. Without loss of generality, suppose also that a is preferred to b and c to d. Then, one would accept that a is better than b weakly more than c is better than d *and* that b is worse than a strictly less than d is worse than c, which is an oxymoron.

Separability, finally, is also an intuitive condition for a preference intensity relation. Suppose $\succsim$ does not satisfy it. Then, $(a,c) \succsim (b,c)$ and $(b,d) \succ (a,d)$ is true for some alternatives in X. With no loss of generality, assume that a and b are preferred to both c and d. Then, $(a,c) \succsim (b,c)$ suggests that a is preferred to c at least as much as b is preferred to c, which in turn indicates a weak preference for a over b (see Claim 3 in Appendix B for a proof). But $(b,d) \succ (a,d)$ similarly suggests that b is preferred to d strictly more than a is preferred to d, which in turn indicates a strict preference for b over a and hence contradicts the above alleged weak preference of a over b.

2.2 Cardinal Utility/Difference Representations

The benchmark model that economists have traditionally used to represent a preference intensity relation –typically under the assumption that both the relation and its domain have a much richer structure than the one above– is that of a *utility-difference/cardinal utility* representation.

Definition 2
*A binary relation $\succsim$ on a set $X \times X$ admits a **utility-difference** representation if there exists a function $u : X \to \mathbb{R}$ such that*

$$(a,b) \succsim (c,d) \iff u(a) - u(b) \geqslant u(c) - u(d). \tag{2}$$

*In addition, $\succsim$ admits a **cardinal utility** representation if such a u exists and is **unique up to a positive affine transformation** in the sense that $v : X \to \mathbb{R}$ also represents $\succsim$ as in (2) if and only if $v = \alpha u + \beta$ for $\alpha > 0$ and $\beta \in \mathbb{R}$.*

Cardinal utility/difference representations of preference intensity relations and their implications for welfare analysis have been at the heart of important historical debates. The reader is referred to Ellingsen (1994), Mandler (1999), Köbberling (2006), Moscati (2013) and Baccelli and Mongin (2016) for some relatively recent formal or historical/methodological analyses, and to Fleurbay and Hammond (2004) and Binmore (2009a) for applications in models that assume interpersonally comparable utilities. While no comprehensive summary of these representations will be attempted here, as a prelude to the model that is introduced later I critically discuss below some aspects of them that are of particular relevance to the present paper's motivation and aims.

The first conceptual concern is that utility-difference representations on finite sets have no simply definable notion of uniqueness.[1] In particular, while they are obviously preserved by positive affine transformations, strictly increasing transformations that are not affine but still deliver a utility-difference representation of the same intensity ordering also exist in general. At the same time, not all strictly increasing transformations preserve a given utility-difference representation.[2] Yet, the fact that such representations on finite sets are not unique up to a positive affine transformation is an obvious drawback of that model, considering that a primary motivating factor for its stronger cardinal-utility counterpart –which is generally applicable on infinite domains– stems from the fact that the utility-difference ratios $\frac{u(a)-u(b)}{u(c)-u(d)}$ are invariant with respect to all transformations that any u which represents the given intensity ordering can afford (in this case, the class of positive affine transformations). This invariance, however, is precisely what enables one to make statements like *"a is preferred to b twice as much as is c is to d"*. Therefore, the fact that this property is lost by utility-difference representations on finite sets raises the question of whether preference intensity orderings on such domains can be represented in an ordinal manner by means of alternative models with simple, precise and intuitive uniqueness properties.

A second concern over utility-difference representations on finite sets is that the characterization of preference intensity relations that admit such representations includes a cancellation axiom that is hard to interpret behaviourally. Specifically, Scott (1964) characterized preference intensity relations that are utility-difference representable in such domains by means of Reversal, Completeness and the requirement (henceforth ***Scott's axiom***) that, for all sequences of elements $a_0, \ldots, a_n, b_0, \ldots, b_n \in X$, and all permutations π and σ of $\{0, 1, \ldots, n\}$, if $(a_i, b_i) \succsim (a_{\pi(i)}, b_{\sigma(i)})$ for all $0 \leqslant i < n$, then $(a_{\pi(n)}, b_{\sigma(n)}) \succsim (a_n, b_n)$. These axioms imply both Transitivity and Separability. Hence, such a preference intensity relation is necessarily basic in the above sense (the converse is not generally true). However, although Scott's axiom is falsifiable with finitely many observations (as also pointed out in Echenique and Saito, 2017), there is no obvious way to summarize and interpret it behaviourally (see also p. 277 in Luce and Suppes, 1965). This axiom is clearly satisfied and can be removed from the characterization of utility-difference representations over finite sets only in the extremely special cases where any two consecutive alternatives in the decision maker's preference ordering are equally spaced in terms of preference intensity (see p. 168 in Krantz et al., 1971). The resulting axiomatization in such cases is indeed interpretable behaviourally (and cardinal uniqueness is, in fact, also obtained), but this comes at a very significant loss of generality.

Yet another conceptual concern that applies to both cardinal utility and utility-difference repre-

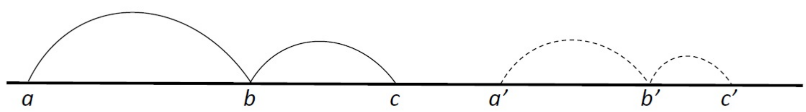
[1] See Krantz et al, (1971, Theorem 2, p. 431).

[2] Both these points are illustrated in Example I, Appendix A.

sentations revolves around the particularly demanding *Concatenation* axiom (sometimes also called *Monotonicity* or the *Sextuple Condition*) that they necessitate. This is stated below and illustrated in Fig. 1.

Concatenation
If $(a,b) \succsim (a',b')$ *and* $(b,c) \succsim (b',c')$, *then* $(a,c) \succsim (a',c')$.

Figure 1. Concatenation
(reproduced from p. 145 in Krantz et al, 1971)



From the point of view of physical distance measurement between earthly objects, Concatenation is clearly a desirable property. It requires, for example, that if the length between points a and b equals that between a' and b', and the length between b and c equals that between b' and c', then a is distanced from c exactly as much as a' is from c'. Yet, its appeal in the context of preference intensity modelling is more questionable. For example, while observing that Concatenation is necessary for a cardinal utility/difference representation, Samuelson (1938b, p. 70) challenged its normative content by noting that *"there is absolutely no a priori reason why the individual's* [preference intensity relation] *should obey this arbitrary restriction"*.

The axiom's appeal is also questionable on descriptive grounds. Although apparently no directly relevant empirical evidence exists, suppose (e.g. as in Suppes and Winet, 1955; Luce and Suppes, 1965) that we are in an environment where a decision maker's preference intensity is indicated by his observable willingness to pay in order to move from one alternative that he is endowed with towards another. Suppose now that suitable data of this kind is available, and suggests that the individual is willing to pay x monetary units to trade away a distasteful mug c for a neutral mug b, and that, once endowed with the latter, he is also willing to trade it for y monetary units to get a mug a with the crest of his alma matter printed on it. Suppose further that the same agent's dataset allows for the Concatenation axiom to be directly tested by also containing observations that suggest he is again willing to pay x in order to trade away a candy bar c' for a cereal bar b', and y to trade the latter for a smoothie a'. It is certainly reasonable to expect that, if he was asked directly how much he would be willing to pay to trade c for a and c' for a', he would respond with some amounts z, z' that would be strictly higher than both x and y. Is it equally reasonable to expect, however, that Concatenation's knife-edge prediction $z = x + y = z'$ will typically be validated in such data?

2.3 Preference Intensity Functions

The model I propose to represent preference intensity relations numerically is not subject to any of the above criticisms of cardinal utility/difference representations, except for the very special cases where they all become equivalent. It also echoes the general view expressed, for example, in Fishburn (1970, p. 82) according to which "[a]*lthough our preference-difference comparisons may not be as* *precise as length comparisons made with precision instruments,* [this is not] *sufficient reason to abandon the idea of such comparisons"*.

Before the model is formally stated below, recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *odd in* $Z \subseteq \mathbb{R}$ if $f(-z) = -f(z)$ is true for all $z \in Z$. For example, $f(z) := z^3$ is odd whereas $g(z) := z^2$ is not.

Definition 3
A binary relation $\succsim$ *on a set* $X \times X$ *is representable by a* ***preference intensity function*** *if there exists an* $s : X \times X \to \mathbb{R}$ *such that, for all* $a, b, c, d \in X$,

$$(a, b) \succsim (c, d) \iff s(a, b) \geqslant s(c, d) \tag{3a}$$

$$s(a, b) = -s(b, a) \tag{3b}$$

$$s(a, b), s(b, c) \geqslant 0 \implies s(a, c) \geqslant s(a, b), s(b, c), \tag{3c}$$

where s *is* ***unique up to an odd and strictly increasing transformation*** *in the sense that* $t : X \times X \to \mathbb{R}$ *also represents* $\succsim$ *as in* (3) *if and only if* $t = f \circ s$ *for some* $f : \mathbb{R} \to \mathbb{R}$ *that is odd and strictly increasing in* $s(X \times X)$.

Property (3a) will be referred to as *order-preservation* and simply asks that preference intensity function s represent a binary relation on the set of ordered *pairs* of alternatives in exactly the same way that an *ordinal* utility function represents a binary relation on the set of *alternatives*. This analogy also motivates the bivariate nature of s. In particular, since the primitive of preference-intensity analysis is the decision maker's attitude towards movements from one alternative to another, and this in turn necessitates analysing the comparisons between pairs of alternatives, a very simple way in which such comparisons can be made numerically is to allow the domains of the intensity-representing function and that of the preference intensity relation to coincide. By doing so, the meaning of the statement "*a is preferred to b more than c is to d*", which is formalized by letting the pair (a, b) lie above (c, d) in the agent's preference-intensity ranking, becomes equivalent to (a, b) being accordingly assigned a higher value than (c, d) by s.

The second property, (3b), is known as *skew-symmetry* and is familiar from decision-theoretic models that allow an individual's simple preference relations to be potentially intransitive (Shafer, 1974; Fishburn, 1982). Its requirement is that, for any two alternatives a and b, s assign the same absolute value to the two pairs (a, b) and (b, a) that contain them, but with the opposite sign. Although imposing skew-symmetry is not necessary for representing a preference intensity relation in an order-preserving way, doing so is associated with at least two advantages. First, under the assumption that s represents $\succsim$ as in (3a), skew-symmetry means that one need only look at the sign of $s(a, b)$ to infer whether the agent's preference relation that is induced by $\succsim$ as in (1) suggests that he strictly prefers a to b (positive sign), if the opposite is true (negative sign), or if he is in fact indifferent between a and b (true if and only if $s(a, b) = s(b, a) = 0$). A second advantage of skew-symmetry is that, in the special case where the underlying preference intensity relation also admits a cardinal/utility-difference representation by means of u, it allows s to be defined by $s(a, b) := u(a) - u(b)$ (see also Example I, Appendix A), and hence to nest that model in a natural way.

The last defining property of a preference intensity function, (3c), will be referred to as *internal consistency*. Under the maintained assumption that s represents $\succsim$ as in (3a), this is interpretable as suggesting that if a is weakly preferred to b and b to c, then a is weakly preferred to c at least as much as a is to b and b is to c. Internal consistency therefore imposes the intuitive restriction that the preference and preference-intensity comparisons are in conceptual harmony in the sense that as the decision maker goes down his preference ranking from a to b and from b to c, his preference intensity between the "remote" alternatives a and c in this ranking is higher than that between the "proximal" alternatives a, b and b, c. In addition, it ensures that the preferences induced by the preference intensity relation represented by s are transitive. Indeed, although (3a) necessitates transitivity of preference intensity comparisons, the joint force of (3a) and (3b) is not sufficient for simple preferences to be transitive as well. Formally, (3a) and (3b) imply that, for all $a, b \in X$, $a \gtrapprox b \Leftrightarrow (a,b) \succsim (b,a) \Leftrightarrow s(a,b) \geqslant 0 \geqslant s(b,a)$, while it then follows from this and (3c) that $a \gtrapprox b \gtrapprox c \Rightarrow s(a,c) \geqslant s(c,a) \Leftrightarrow (a,c) \succsim (c,a) \Leftrightarrow a \gtrapprox c$.

In line with the motivating remarks in this direction that have been made so far, and in sharp contrast to the properties of cardinal-utility functions in their respective domains, preference intensity functions are essentially ordinally unique, independently of whether their domain is a finite set –as will be assumed in this paper– or a suitably structured infinite set that supports a cardinal utility representation of some preference intensity relation. Also in contrast with utility-difference representations on finite sets whose uniqueness properties lie somewhere between strictly increasing and positive affine transformations but with there being no simple way to describe them, the only difference between the standard notion of ordinal uniqueness and the one associated with preference intensity functions is that, in addition to being strictly increasing in the relevant domain, the transformation f must also be an *odd* function, hence to also satisfy the simple extra condition $f(-z) = -f(z)$ in that domain. Imposing the latter property is necessary to ensure that, along with its order-preservation and internal-consistency properties which are maintained by strictly increasing transformations, a preference intensity function's skew-symmetry is also preserved in the process of transforming it into another function that represents the same intensity ordering. For example, the non-linear transformation $f(z) := z^3$ belongs to this class, whereas the positive affine transformation $g(z) := 2z + 2$ does not. Let it be remarked, finally, that while uniqueness up to odd and strictly increasing transformations is apparently a novel concept in the extensive measurement literature, it may be thought of as the ordinal analogue of the well-known class of positive *linear* transformations (see below).

Theorem 1
The following are equivalent for a binary relation $\succsim$ on a finite set $X \times X$:

1. *$\succsim$ is a basic preference intensity relation on X.*
2. *$\succsim$ is representable by a preference intensity function $s : X \times X \to \mathbb{R}$.*

The class of preference intensity orderings on finite sets that are representable by preference intensity functions is therefore characterized by the Weak Order, Reversal and Separability axioms. That these axioms are necessary for such a representation is straightforward. The argument in the part of the proof that establishes how these three axioms are also sufficient goes as follows. First, it is shown that the induced preference relation of such an intensity ordering is complete and transitive. Then, the choice set is partitioned into indifference classes that are linearly ordered by the induced strict preference relation. Next, the collection of pairs (a, b) is considered where $a \ggcurly b$, and a skew-symmetric function s is constructed such that, for any such pair (a, b), $s(a, b)$ is the number (augmented by 1) of intensity-equivalence classes that consist of pairs of this kind that lie strictly below –according to the strict preference intensity relation– the intensity-equivalence class where (a, b) belongs to. The bulk of the proof shows that this s is indeed a preference intensity function that represents the underlying $\succsim$. A noteworthy implication of the above constructive argument is the following.

Corollary 1
A basic preference intensity relation on a finite set X is representable by a preference intensity function s whose range is a set of consecutive integers:

$$s(X \times X) = \{-k, \ldots, -1, 0, 1, \ldots, k\}. \tag{4}$$

To illustrate how preference intensity functions are intuitively less demanding than utility functions that support cardinal/utility-difference representations, and how they are also associated with improved explanatory power, let us revisit the example given earlier when Concatenation was critically evaluated. In that example, the data that was assumed to be available suggest that preference intensity comparisons between the choice objects are such that $(a, b) \sim (a', b')$, $(b, c) \sim (b', c')$ and $(a, b) \succ (b, a)$, $(a', b') \succ (b', a')$. Assuming also that the amounts z and z' that were paid to move from c to a and from c' to a', respectively, satisfy $z, z' > x, y$, this data also suggests $(a, c) \succ (a, b), (b, c)$ and $(a', c') \succ (a', b'), (b', c')$. These comparisons are easily representable by a preference intensity function s on $X := \{a, b, c, a', b', c'\}$. Importantly, however, and in sharp contrast to cardinal utility/difference representations, this model can still represent the agent's preference intensity ordering even if $(a, c) \nsim (a', c')$, which would arise in all cases where the amounts z and z' differ. More generally, if x is spent to move from b to a and y is spent to move from c to b, then the above models predict that $z = x + y$ would be spent to move from c to a, whereas the more general preference intensity model predicts that the amount z is simply greater than both x and y. Example II in Appendix B (due to Köbberling, 2006) provides a concrete basic preference intensity relation that violates Concatenation (hence is not utility-difference representable) but nevertheless admits a representation by a preference intensity function.

The concept defined next lies at the heart of the relationship between preference-intensity and utility-difference representations, and, as shown below, clarifies how the former generalize the latter.

Definition 4
A binary relation $\succsim$ on a set $X \times X$ is represented by a ***triangularly additive*** *preference intensity function $s : X \times X \to \mathbb{R}$ if, for all $a, b, c \in X$,*

$$s(a, c) = s(a, b) + s(b, c). \tag{5}$$

In addition, such an s is ***unique up to a positive linear transformation*** *whenever $t : X \times X \to \mathbb{R}$ is also triangularly additive and represents $\succsim$ if and only if $t = \alpha s$ for some $\alpha > 0$.*

Triangular additivity is generally not satisfied by preference intensity functions, and both sub-additive and super-additive deviations from this property generally occur within the context of the *same* such representation (see Examples I, II in Appendix A). When this condition is satisfied, however, it implies both (3b) and (3c). Triangularly additive preference intensity functions are therefore characterized by (3a) and (5) only. To clarify the uniqueness property that such representations *might* afford under suitable conditions on both X and $\succsim$ (see also below), note that, although any odd and strictly increasing transformation of s would still represent the same $\succsim$, uniqueness up to positive linear transformations means that triangular additivity would be lost by all such transformations that are not of this kind.

Corollary 2

The following are equivalent for a binary relation $\succsim$ on a finite set $X \times X$:

1. *$\succsim$ satisfies Completeness, Reversal and Scott's Axiom.*
2. *$\succsim$ is utility-difference representable.*
3. *$\succsim$ is representable by a triangularly additive preference intensity function.*

This result –which is illustrated in the second part of Example I, Appendix A– shows that, in finite domains that are of interest in this paper, the proposed preference intensity model includes the utility-difference model as a special case where the underlying preference intensity order is representable by a preference intensity function s that takes the special *additively separable* form $s(a, b) \equiv u(a) - u(b)$. The equivalence between the first two statements is due to Scott, (1964, Theorem 3.2). With regard to $3 \Rightarrow 2$, it is well-known (see, for example, Theorem 2, p. 356 in Aczél, 1966 or pp. 97-98 in Falmagne, 2002) that, under very general conditions which encompass those of Corollary 2, the solution to a so-called *Sincov* functional equation $f(x, y) = f(x, z) + f(z, y)$ is given by $f(x, y) = g(x) - g(y)$ for a unique function g. For completeness, a simple direct proof is provided nonetheless. In view of the above remarks, finally, the converse implication $2 \Rightarrow 3$ is obvious.

As already discussed, given the finiteness of the domain, the utility-difference representation that is postulated in the second statement of Corollary 2 has no simply definable uniqueness properties. As far as the third statement is concerned, while a preference intensity function that represents a given $\succsim$ on such a set is unique up to an odd and strictly transformation (Theorem 1), the uniqueness properties of *triangularly additive* representations of this kind are analogously complex and depend on those of the corresponding utility-difference representations (see also below).

Importantly, in the process of showing that the assumptions in Lange (1934) were not sufficient to guarantee the existence of a cardinal utility representation in the case where $X = \mathbb{R}^n_+$, Samuelson (1938b) established a cardinal version of the implication $3 \Rightarrow 2$ by employing a differentiability-based argument. Specifically, Samuelson considered a bivariate function G that satisfied (3a) and (3b) and aimed at identifying the conditions under which a "cardinal"[3] (unique up to a positive

[3]Samuelson (1938b) used the term "positive linear transformation" for what is now commonly referred to (and in the present paper too) as a positive *affine* transformation, and identified the term "cardinal" with this uniqueness property. For the function

linear transformation) G of this kind could be written as $G(x, y) = u(x) - u(y)$ for some u that itself is unique up to a positive *affine* transformation. Using this paper's terminology, Samuelson's answer was that G must be triangularly additive [his condition (15)]. It is worth emphasizing, however, that Samuelson treated G only as an instrument towards disproving the claim made in Lange (1934). In particular, he did not suggest that triangular additivity be weakened as in (3c), and neither did he motivate his function G as a potentially more general way to represent preference intensity relations.

Moreover, Samuelson's equivalence between such bivariate and univariate mappings was axiomatized and extended to a more general class of domains in Krantz et al., (1971, Theorem 1, p. 147). There, it was also pointed out that if $\succsim$ is represented by a bivariate s that is unique up to a positive linear transformation and also by a univariate u that is unique up to a positive affine transformation, then t and v also represent $\succsim$ as in s and u, respectively, if and only if $t = \alpha s$ and $v = \alpha u + \beta$ for $\alpha > 0$ and $\beta \in \mathbb{R}$, i.e. t and v necessarily share the same multiplicative constant (the latter is implicit in the statement of that result). Going back now to the case of finite domains and Corollary 2, the way in which the uniqueness properties of a triangularly additive preference intensity representation of $\succsim$ was claimed above to depend on those of a utility-difference of $\succsim$ is by analogy with the way in which their cardinal counterparts do so in their respective domains.

3 Revelation

3.1 Choices and Foregone Resources

Since the defining works of Samuelson (1938a), Richter (1966) and Afriat (1967) (see Chambers and Echenique, 2016 for a thorough recent treatment of this literature), choice datasets in revealed preference theory have typically comprised collections of pairs of general menus or price-generated competitive budget sets, and the choice(s) observed at these menus. As is well known, however, even when such datasets are rationalizable by an ordinal utility function, the information contained in them does not make it possible for the analyst to make any inferences about the decision maker's intensity of preference over the alternatives in question. Yet, consistent also with casual empiricism and introspection, it has been argued in the literature that an individual could in certain contexts reveal his preference intensities in a way that is *in principle* observable to a third party.

As mentioned earlier, a natural example of the means through which such revelation could take place is the amount of money one is willing to pay (alternatively, work he is prepared to do, distance to travel, etc.) in order to choose something. In particular, the experimental method that was proposed by Suppes and Winet (1955) builds on the idea of endowing subjects with choice alternatives b and d and asking them how much money they would be willing to pay in order to trade them with a and c, respectively. If this amount is strictly higher for a than for c, then the analyst could make the inference $(a, b) \succ (c, d)$. Although the authors proposed this design as a means towards testing conformity of a subject's behaviour with the cardinal utility/difference model, it is obviously generally applicable for recovering a subject's preference intensity ordering, even if the latter does

G in Samuelson (1938b), however, the correct interpretation of the term "cardinal" on p. 68 is uniqueness up to a positive *linear* transformation in the modern sense of uniqueness up to multiplication by a positive constant (also adopted in the present paper).

not admit such a representation.[4]

Another variable of increasing interest in behavioural economics and revealed-preference analysis that in some cases could also be thought of as a proxy for preference intensity –and which is generally relevant in both lab and field/market environments– is the time it takes a decision maker to make a choice from a menu. In this literature, response-time data are typically generated from binary menus and, assuming implicitly that the individual's preferences over the underlying alternatives are weakly ordered, the claim is that shorter response times are associated with easier choices and hence, intuitively, with a higher preference-intensity distance between the feasible alternatives compared to cases where response times are higher (Krajbich, Oud, and Fehr, 2014; Konovalov and Krajbich, 2017; Echenique and Saito, 2017).

While different in fundamental ways, what the above variables have in common is that they represent a *scarce resource* that the decision maker forgoes while in the process of choosing an alternative. The revealed-preference foundations that are given below for the general preference intensity model that was introduced earleir builds on the premise that such forgone resources are often observable, and treats *resource* as an umbrella term that encompasses any relevant variable that could act as a reliable proxy for a decision maker's preference intensity in a given choice environment.

A formal definition of the kind of data that the analyst is assumed to have access to is given next.

Definition 5
A ***behavioural dataset***

$$\mathcal{D} = \big(A_i, C(A_i), r_i\big)_{i=1}^{k}$$

on a finite set X is a collection of data points $(A_i, C(A_i), r_i)$ that consist of a menu $A_i \subseteq X$ of feasible alternatives; the set $C(A_i) \neq \varnothing$ of alternatives that are choosable in A_i; and the amount $r_i \in \mathbb{R}$ of some observable resource that the decision maker has foregone in order to choose from A_i.

The following derived notation will also be helpful.

$$\begin{array}{rcl} \mathcal{D}_x & := & \{x \in X : x = x_i \text{ for some } i \leqslant k\} \\ \mathcal{D}_A & := & \{A \subseteq X : x_i \in C(A) \text{ for some } i \leqslant k\} \\ \mathcal{D}_r & := & \{r \in \mathbb{R} : r = r_i \text{ for some } i \leqslant k\} \end{array}$$

A few remarks are due. The above definition of a behavioural dataset requires the agent to have made an active choice at each menu but, importantly, allows for more than one alternative to be chosen at any such feasible set. It therefore encompasses the standard case where a single chosen option is observed in each menu, while it remains operational, for example, in cases where either more than one option was actually observed to be chosen, or where the analyst had at their disposal some additional information (e.g. from stated-preference data) that allowed for any indifference the decision maker had expressed to be incorporated in such a way that the observed single-valued choices

[4] Notice that, although money plays an important role in the neoclassical theory of demand, revealed-preference models in this context, which take pairs of prices and demanded bundles as their primitive dataset, essentially *have* to build on the assumption that all income is always spent at each such observation in order to provide foundations for recovering a locally non-satiated (typically, strictly monotonic) preference relation over consumption bundles. This in turn leaves no role for spent money in these models to potentially act as an intensity-revealing resource.

are convertible into possibly multi-valued ones. Finally, a behavioural dataset also allows for the possibility that the same menu was presented to the decision maker more than once, with both his choice and resource value potentially varying across the instances where this happened. Therefore, a third motivation for the possibly multi-valued nature of choices in a behavioural dataset is that, for any menu $A \in \mathcal{D}_A$, the set $C(A)$ stands for the collection of all alternatives that were chosen (one at a time) whenever the decision maker was faced with menu A.

Next, for any $x \in \mathcal{D}_x$ let

$$R_x := \{r_i \in \mathcal{D}_r : x \in C(A_i) \text{ for some } i \leqslant k\}$$

denote the collection of all resource values that were allocated to x when this alternative was chosen in some menu(s) in $\mathcal{D}_A$. This notation highlights another crucial feature of behavioural datasets whereby the *same* alternative x can be chosen at different menus A_i, A_j in $\mathcal{D}_A$ and *distinct* resource levels $r_i \neq r_j$ are foregone in that process. This feature is relevant, for example, in a context where it is appropriate to use the decision maker's response times as a proxy for his underlying preference intensity between alternatives. In such cases it is reasonable that the same alternative x could be chosen as the most preferred one in menu A_i with two options and in a menu A_j with seven options while the response time is higher in the latter case. But even when the number of alternatives is the same (often fixed at two), the inherently noisy nature of the response-times variable suggests that it is unlikely that exactly the same number of seconds (even rounded up to the first decimal point) will be associated with the same choice at the same menu on different instances.

In general, the proposed notion of a behavioural dataset allows for treating as separate the decision maker's choice of a feasible option and the resource amount that he sacrifices for it. An alternative specification would be to enrich the choice set by defining it as a collection of pairs of options and resource amounts that might be foregone when these options are chosen. While this approach is not orthogonal to the one adopted here (choices and response times are a case in point), resource amounts in a behavioural dataset are generally allowed to be derivable from a process that is distinct from the one that provides data on choices over alternatives (e.g. an experiment that features a choice part and a valuation part over the same set of options). In such cases, it is to be understood that this dual decision data is revealed by the individual in a context where, even when some time has elapsed between the point when a choice of an alternative was made and when the corresponding resource amount was allocated, the decision maker's preferences and preference intensities remained unaltered. This requirement, in turn, is essentially a stronger version of the assumption that underpins classical revealed preference models, whereby choices between the same two alternatives at distinct but "proximal" time points are implicitly also assumed to have been derived by the same underlying preferences. In any case, while this is an important methodological issue that warrants further investigation, a favourable aspect of the proposed modelling approach is its notational simplicity and analytical convenience.

3.2 Preference-Intensity Rationalizable Datasets

Some further notation is laid out next before the model of revealed preference intensity is introduced. Specifically, given two sets of real numbers A and B, define their algebraic difference $A - B$ by

$$A - B := \{a - b \in \mathbb{R} : a \in A,\ b \in B\}.$$

For example, if $A = \{4, 6\}$ and $B = \{3, 5\}$, then $A - B = \{-1, 1, 3\}$.

Definition 6
*A behavioural dataset $\mathcal{D}$ on a finite set X is **preference-intensity rationalizable** if there exists a function $s : X \times X \to \mathbb{R}$ such that, for all $A \in \mathcal{D}_A$ and all $a, b, c, d \in \mathcal{D}_x$,*

$$a \in C(A) \iff s(a, z) \geqslant 0 \text{ for all } z \in A \tag{6a}$$

$$s(a, b) \geqslant s(c, d) \iff \min\{R_a - R_b\} \geqslant \max\{R_c - R_d\}. \tag{6b}$$

The first part of this notion of rationalizability, (6a), reformulates in the present setting what is standard in the rational revealed-preference literature: for every menu in the dataset, the alternative chosen at this menu is a most preferred feasible one according to a stable, complete and transitive simple preference relation. The second requirement, (6b), is novel and reflects the preference-intensity component that is revealable through such datasets. According to it, given four alternatives a, b, c, d that were observed to be chosen in $\mathcal{D}$, a is revealed preferred to b weakly more than c is to d if and only if the difference between the observed resources allocated to a and b is always weakly higher than the difference in the resources allocated to c and d. At a formal level, this robustness requirement of preference-intensity rationalizability is dictated by the skew-symmetry of s. The latter requires, in particular, that the expression to the right of the equivalence sign in (6b) also feature two skew-symmetric terms. This is indeed the case here since, for any $a, b \in X$,

$$\min\{R_a - R_b\} = (-1) \cdot \max\{R_b - R_a\}.$$

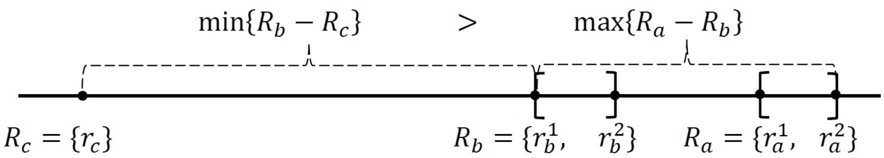
On conceptual grounds, this kind of robustness has some desirable features in the present environment where the decision maker may forgo different resource amounts when choosing the same alternative at different menus. Consider, for example, the less demanding revelation criterion whereby $s(a, b) \geqslant s(c, d)$ if and only if $\max\{R_a - R_b\} \geqslant \max\{R_c - R_d\}$ and $\min\{R_a - R_b\} \geqslant \min\{R_c - R_d\}$ are both true. Unlike (6b), this allows for the possibility where, despite the fact that the set $R_a - R_b$ lies weakly to the right of the set $R_c - R_d$ even when both of these inequalities are strict, there still exist $\overline{r} \in (R_a - R_b)$ and $\widetilde{r} \in (R_c - R_d)$ such that $\overline{r} < \widetilde{r}$. Thus, this weaker notion does not rule out counterintuitive cases where there is at least one combination of resource-allocation data points in $\mathcal{D}$ which suggests that c is preferred to d more than a is to b, even though all other data points suggest that the opposite is true. The proposed model therefore formalizes the view that when such conflicting evidence exists on the observed foregone-resource values, no preference-intensity inference can be made on the alternatives involved in this conflict.

Notice now that, under (6), a is revealed preferred to b *strictly more* that c is to d –denoted by $(a,b) \succ^R (c,d)$– if and only if the difference $r_a - r_b$ is weakly higher than $r_c - r_d$ under all observed resource amounts for these alternatives *and strictly higher* under some combination of these observed amounts. Equivalently,

$$(a,b) \succ^R (c,d) \iff \begin{cases} \min\{R_a - R_b\} \geqslant \max\{R_c - R_d\} \\ \& \\ \max\{R_a - R_b\} > \max\{R_c - R_d\} \end{cases} \tag{7}$$

Fig. 2 provides a graphical example that illustrates how the relation $\succ^R$ becomes operational through (7).

Figure 2. Example where $(b,c) \succ^R (a,b)$



Moreover, (6) implies that a is revealed preferred to b *exactly as much* as c is preferred to d – denoted by $(a,b) \sim^R (c,d)$– if and only if the difference in the resource amounts between the elements of the two pairs is always the same. Formally,

$$(a,b) \sim^R (c,d) \iff \begin{cases} \min\{R_a - R_b\} \geqslant \max\{R_c - R_d\} \\ \& \\ \min\{R_c - R_d\} \geqslant \max\{R_a - R_b\} \end{cases} \tag{8}$$

$$\iff R_a - R_b = R_c - R_d = \{\overline{r}\}, \qquad \overline{r} \in \mathbb{R}$$

$$\implies |R_a| = |R_b| = |R_c| = |R_d| = 1.$$

Therefore, unlike when a is revealed preferred to b strictly more than c is to d, the condition $r_a - r_b = r_c - r_d$ for all such resource values which characterizes the case $(a,b) \sim^R (c,d)$ actually necessitates that the same alternative-specific resource value r_x always be allocated to every $x \in \{a,b,c,d\}$.

Let us now turn to the conditions on behavioural datasets that are both necessary and sufficient for the latter to be rationalizable by a preference intensity function in the sense of (6). Before stating the axioms, define the binary relations of direct revealed weak preference, strict preference and indifference $\succsim^R$, $\succ^R$ and $\approx^R$ on X by

$$x \succsim^R y \iff x \in C(A),\ y \in A \quad \text{for some } A \in \mathcal{D}_A,$$

$$\begin{aligned} x \succ^R y &\iff x \succsim^R y \text{ and } y \not\succsim^R x \\ &\iff x \in C(A),\ y \in A \backslash C(A) \quad \text{for some } A \in \mathcal{D}_A, \end{aligned}$$

$$x \approx^R y \iff x \succsim^R y \text{ and } y \succsim^R x,$$

respectively.

Cross-Modal Consistency (CMC)
If $a \succsim^R b$ *(resp.* $a \succ^R b$*) and* $b \in \mathcal{D}_x$*, then* $\min R_a \geqslant \max R_b$ *(resp.* $\min R_a > \max R_b$*).*

CMC requires that the decision maker's choices with respect to any two alternatives that were observed to be chosen in $\mathcal{D}$ never contradict the way in which resources were allocated to these alternatives. As such, it rules out preference-reversal phenomena whereby the same individual ranks x and y differently depending on whether the ranking is elicited through choice or valuation. In particular, even though x and y can have different resource amounts allocated to them at different menus in which they are chosen, the axiom requires that the act of choosing x in the presence of (resp. *over*) y in some menu impose some structure on these allocations by preventing any resource amount that has been allocated to y to ever strictly (resp. *weakly*) exceed any amount that has been allocated to x. An easily verifiable consequence of this requirement is that choices alone conform to the Strong Axiom of Revealed Preference (SARP): $x_1 \in C(A_1)$, $x_2 \in A_1 \backslash C(A_1)$, $\ldots$, $x_{m-1} \in C(A_{m-1})$, $x_m \in A_m \backslash C(A_{m-1})$ and $x_m \in C(A_m)$ implies $x_1 \notin A_m$.

Resource-Difference Comparability (RDC)
If $a \succsim^R b$, $c \succsim^R d$ *and* $b, d \in \mathcal{D}_x$*, then* $\min\{R_a - R_b\} \geqslant \max\{R_c - R_d\}$ *or* $\min\{R_c - R_d\} \geqslant \max\{R_a - R_b\}$.

This axiom requires that the differences in resource amounts that are allocated to the elements of any two pairs of alternatives that were observed to be chosen in $\mathcal{D}$ never change sign. Specifically, if the difference between *any* resource amounts allocated to x and y is ever (weakly or strictly) greater than *some* resource difference corresponding to w and z, then it is greater (generally weakly) to *all* observed resource allocations that pertain to these four alternatives. An analogy can be drawn between RDC and the Weak Axiom of Revealed Preference (WARP). In its general choice-theoretic formulation, WARP requires that if x is ever chosen over y, then y is never choosable in the presence of x. This implies that the revealed strict preference relation $\succ^R$ is asymmetric. Notice now that a revealed strict preference intensity relation S could similarly be defined by $(x, y)S(w, z)$ whenever there exist r_x, r_y, r_w, r_z such that $r_x - r_y > r_w - r_z$. RDC would then require that this S be asymmetric. In particular, although RDC allows for $r_x - r_y > r_w - r_z$ and $r'_x - r'_y = r'_w - r'_z$ to both be true for different combinations of resource allocations on these four alternatives, RDC requires that any strict inequality sign that is observed between them never be reversed.

Theorem 2
The following are equivalent for a behavioural dataset $\mathcal{D}$:

1. *$\mathcal{D}$ satisfies Cross-Modal Consistency and Resource-Difference Comparability.*
2. *$\mathcal{D}$ is preference-intensity rationalizable.*

As is the case with many revealed-preference characterizations, it is straightforward that CMC and RDC are necessary for a dataset to be rationalizable by a preference intensity function, although establishing that they are also sufficient is less obvious. The general direction of the proof follows the standard approach whereby a certain incomplete binary relation that is defined in terms of observable data is completed in a suitable way. A key difference between standard applications of this technique and the one required by the current problem is that the binary relation in question (i.e. the revealed preference intensity relation) is defined on $X \times X$ rather than on X. In addition, this primitive partial relation must be defined in such a way that: i) it possesses all the properties of a basic preference intensity relation except completeness; ii) it affords a completion by a *basic* preference intensity relation; c) it induces the (partial) revealed preference relation that is defined in the standard way (see above). The proof provides a construction that achieves these objectives.

Since, as shown in Corollary 2, a utility-difference representation comes about as a special case of a representation by means of a preference intensity function, it is natural to also study the way in which the behavioural implications of the corresponding classes of preference intensity orderings compare. In fact, under the maintained assumption that the analyst has access to behavioural datasets as defined above, one can go beyond that and define a falsifiable notion of *cardinal-utility rationalizable* datasets by imposing a sufficiently strong structure on the observables.

Definition 7
A behavioural dataset $\mathcal{D}$ on a finite set X is ***cardinal-utility rationalizable*** *if there exists a function $u : X \to \mathbb{R}$ such that, for all $i \leqslant k$,*

$$a_i \in C(A_i) \iff u(a_i) \geqslant u(z) \text{ for all } z \in A_i \tag{9a}$$

$$u(a_i) = \alpha r_i + \beta, \qquad \text{for some } \alpha > 0 \text{ and } \beta \in \mathbb{R}. \tag{9b}$$

Notice that (9b) here implies that $\mathcal{D}$ is cardinal-utility rationalizable only if the following is satisfied:

Resource Precision (RP)
If $a \in \mathcal{D}_x$, then $R_a = \{r_a\}$.

The proposed notion of cardinal-utility rationalizability is therefore consistent both with the standard requirement that the decision maker's choice behaviour be rationalizable by some utility function, and also with the idea that, in an environment where additional data is available on some resource variable that may be thought of as a reliable proxy for the decision maker's intensity of preference, the rationalizing utility function should indeed be cardinal with respect to this variable.

To verify that the CMC and RDC axioms are also implied by (9), notice that, for given $\alpha > 0$ and $\beta \in \mathbb{R}$,

$$\begin{aligned} u(a) - u(b) \geqslant u(c) - u(d) &\iff (\alpha r_a + \beta) - (\alpha r_b + \beta) \geqslant (\alpha r_c + \beta) - (\alpha r_d + \beta) \\ &\iff r_a - r_b \geqslant r_c - r_d \\ &\iff \min\{R_a - R_b\} \geqslant \max\{R_c - R_d\} \\ &\iff s(a,b) \geqslant s(c,d), \end{aligned}$$

where $s(a,b) := u(a) - u(b)$ is the triangularly additive preference intensity function that is induced by u. Thus, (9) is indeed a special case of (6). Moreover, since RP implies $\min\{R_a - R_b\} = \max\{R_a - R_b\} = R_a - R_b$ for all $a, b \in \mathcal{D}_x$, RDC is automatically satisfied.

Corollary 3
The following are equivalent for a behavioural dataset $\mathcal{D}$ on a finite set X:

1. *$\mathcal{D}$ satisfies Cross-Modal Consistency and Resource Precision.*

2. *$\mathcal{D}$ is cardinal-utility rationalizable.*

The simple argument above shows that $2 \Rightarrow 1$ is true in this result. The converse implication is also established by proceeding along the lines of the proof of Theorem 2 and setting $u(a) = r_a$ for all $a \in \mathcal{D}_x$ and $u(a') = r$ for all $a' \in X \backslash \mathcal{D}_x$, where $r \in \mathbb{R}$ is sufficiently low.

4 Related Literature

As discussed in Binmore (2009b, Section 4.3), for example, the von Neumann and Morgenstern (1944) expected utility model has often been interpreted as one that allows for a decision maker's intensity of preference to be revealed by his choices over risky objects. The claim here is that if the agent prefers a over b and c over d, then the former preference would be more intense than the latter if he preferred lottery $M = \left(\frac{1}{2} \circ a;\ \frac{1}{2} \circ d\right)$ to $L = \left(\frac{1}{2} \circ b;\ \frac{1}{2} \circ c\right)$. The argument in support of this claim is that, under expected utility, M is preferred to L if and only if $u(a) - u(b) > u(c) - u(d)$, where u is unique up to a positive affine transformation. The argument against it –articulated, for example, in Luce and Raiffa (1957, p. 32), Blume (2010, p. F160), Baccelli and Mongin (2016, p. 281) and, in more detail and formality, Ellingsen (1994, pp. 133-138)– might be summarized as positing that this is essentially the only thing that the expected utility and cardinal utility models *generally* have in common. Specifically, the two models build on distinct axiomatic systems that apply to different domains and, as a result, the von Neumann-Morgenstern utility function u does not generally allow for recovering a preference intensity ordering on the set of final outcomes that is representable by a cardinal utility function u in the sense of (2).

In the stochastic choice literature, taking as primitive the choice probabilities $p(a,b)$ from all binary menus $\{a,b\}$ that are derived from a suitably structured infinite set of alternatives, Debreu (1958) identified necessary and sufficient conditions on these for the representation $p(a,b) \geqslant p(c,d) \Leftrightarrow u(a) - u(b) \geqslant u(c) - u(d)$ to be obtained for some u that is unique up to a positive affine transformation. Debreu (1958) further suggested that $p(a,b) > p(c,d)$ in this model be read as "a is preferred to b more than c is to d". However, as Baccelli and Mongin (2016, p. 282) pointed out, such a preference-intensity interpretation of choice probabilities is not satisfactory on intuitive grounds given the fact that the postulated stochasticity of the available data runs against the possibility of the decision maker behaving like an ordinal utility maximizer (except in the extreme 0-1 probability cases), let alone one that has a stable and cardinal-utility representable preference intensity ordering. Phrased more generally, since a cardinal representation is normally expected to refine a pre-existing ordinal representation, it is unsatisfactory to propose a cardinal representation by lack of an ordinal one, and with the former even dissolving when the latter obtains.[5]

Assuming an infinite dataset that comprises single choices from binary menus and the response times associated with them, and assuming directly a cardinal utility representation on this set (specifically, Theorem 1, p. 147, Krantz et al, 1971), Echenique and Saito (2017) recently provided conditions on such a dataset that characterize the model in which $x \ggg^B y \Leftrightarrow u(x) > u(y)$ and $u(x) - u(y) = f\big(t(x,y)\big)$, where $x \ggg^B y$ here means that x is chosen over y in the binary menu $\{x,y\}$, $t(x,y)$ is the response time for choosing x over y, f is a strictly monotonic function, u is a cardinal utility function, and u and f are jointly unique up to a positive affine transformation and up to a positive linear transformation, respectively, in the sense that they share the same multiplicative constant. The authors also provided versions of the above model in the testable cases of finite datasets of this kind. Importantly, however, the above strong uniqueness property of the pair (u,f) is then lost.

The Echenique and Saito (2017) model encompasses the case where f is strictly decreasing, and hence provides a formal explanation of relevant experimental evidence (see, for example, Krajbich, Oud, and Fehr, 2014; Konovalov and Krajbich, 2017 and references therein) suggesting that a higher preference intensity is often associated with a shorter response time. Notably, response times in that model satisfy $(a \ggg^B b, b \ggg^B c) \Rightarrow t(a,c) < t(a,b), t(b,c)$, which is analogous to the internal-consistency property (3c) of preference intensity functions. Moreover, replacing the utility-difference representation in the finite-data version of the Echenique-Saito model with a general preference intensity representation leads to the more general version of their model whereby $x \ggg^B y \Leftrightarrow s(x,y) > 0$ and $s(x,y) = f\big(t(x,y)\big)$. Independently of this particular formulation, however, the model of preference-intensity rationalizability that is laid out in (6) can also formalize the above interpretation of response times when $\mathcal{D}$ consists of binary menus and the associated choices and response times, by letting the resource-value set R_x for each $x \in \mathcal{D}_x$ comprise the collection of response times where x was observed to be chosen, multiplied by -1 in order to reflect the postulated negative relationship between decision time and preference intensity.

Finally, distinct from but conceptually related to the problems studied in this paper is Mandler's (2006) focus on the identification of classes of transformations that preserve certain properties of utility functions (e.g. *concavity* as opposed to mere *quasi-concavity*). That paper's primitive is a collection of utility functions –called *"psychologies"*– that represent some simple preference ordering over some domain. At the two ends of the spectrum of this concept lie those collections that comprise all strictly increasing and all positive affine transformations of a given utility function, respectively.

[5] I thank Jean Baccelli for suggesting this sharper statement.

The class of all concave utility functions that represent a given preference relation, for example, is formally shown to correspond to a case in between these two extremes. One could similarly think of the uniqueness properties of some utility-difference representation of a given preference-intensity ordering on a finite set as such a collection that also lies somewhere in between these two polar cases.

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Appendix A: Examples

Example I.

Suppose $X = \{x_1, x_2, x_3, x_4\}$ and let the binary relation $\succsim$ on $X \times X$ be such that

$$\begin{aligned}
(x_1, x_4) &\succ (x_1, x_3) \\
&\succ (x_1, x_2) \sim (x_2, x_4) \\
&\succ (x_3, x_4) \\
&\succ (x_2, x_3) \\
&\succ (x_1, x_1) \sim (x_2, x_2) \sim (x_3, x_3) \sim (x_4, x_4) \\
&\succ (x_3, x_2) \\
&\succ (x_4, x_3) \\
&\succ (x_2, x_1) \sim (x_4, x_2) \\
&\succ (x_3, x_1) \\
&\succ (x_4, x_1)
\end{aligned}$$

This $\succsim$ is a basic preference intensity relation that induces the linear order $\gtrapprox$ on X where

$$x_1 \ggcurly x_2 \ggcurly x_3 \ggcurly x_4.$$

Moreover, $\succsim$ can be represented by the preference intensity function $s : X \times X \to \mathbb{R}$ defined by

$$\begin{aligned}
s(x_1, x_4) &= 5 \\
s(x_1, x_3) &= 4 \\
s(x_1, x_2) = s(x_2, x_4) &= 3 \\
s(x_3, x_4) &= 2 \\
s(x_2, x_3) &= 1 \\
s(x_1, x_1) = s(x_2, x_2) = s(x_3, x_3) = s(x_4, x_4) &= 0 \\
s(x_i, x_j) &= -s(x_j, x_i).
\end{aligned}$$

Now observe that, since $s(x_1, x_4) = 5 \neq 4 + 2 = s(x_1, x_3) + s(x_3, x_4)$, s is not triangularly additive. However, defining $f : \mathbb{R} \to \mathbb{R}$ by

$$f(z) := \begin{cases} z, & \text{if } |z| \leqslant 4 \\ z + 1, & \text{if } z > 4 \\ z - 1, & \text{if } z < -4 \end{cases}$$

results in the odd and strictly increasing –in $s(X \times X)$– transformation $t : X \times X \to \mathbb{R}$ that also represents $\succsim$ and is such that

$$t(a,b)=\begin{cases} s(a,b), & \text{if } (a,b)\neq(x_1,x_4),(x_4,x_1)\\ 6, & \text{if } (a,b)=(x_1,x_4)\\ -6, & \text{if } (a,b)=(x_4,x_1),\end{cases}$$

hence triangularly additive. Moreover, $s=g\circ t$ for $g:\mathbb{R}\to\mathbb{R}$ defined by

$$g(z):=\begin{cases} z, & \text{if } |z|<6\\ z-1, & \text{if } z\geqslant 6\\ z+1, & \text{if } z\leqslant -6,\end{cases}$$

where g is also odd and strictly increasing in $t(X\times X)$.

Now, since t is triangularly additive, and consistent with Corollary 2, there exists $u:X\to\mathbb{R}$ such that $t(a,b)\equiv u(a)-u(b)$, where u is a utility-difference representation of $\succsim$ that is recovered by solving the linear system

$$\begin{pmatrix} 1 & 0 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 1 & -1 & 0\end{pmatrix}\begin{pmatrix} u(x_1)\\ u(x_2)\\ u(x_3)\\ u(x_4)\end{pmatrix}=\begin{pmatrix} 6\\ 4\\ 3\\ 3\\ 2\\ 1\end{pmatrix},$$

which yields

$$u(x_1)=6,\qquad u(x_2)=3,\qquad u(x_3)=2,\qquad u(x_4)=0.$$

The fact that this utility-difference representation of $\succsim$ is not cardinal can be verified by considering the transformation $v=h\circ u$ that is strictly increasing in $u(X)$ and piecewise-affine (but not affine) defined by

$$v(x_1)=21,\qquad v(x_2)=12,\qquad v(x_3)=8,\qquad v(x_4)=3$$

and derived, for example, by $h:\mathbb{R}\to\mathbb{R}$ such that

$$h(z):=\begin{cases} 3.5z, & \text{if } z\geqslant 6\\ 4z, & \text{if } z\in(1,6)\\ z+3, & \text{if } z\leqslant 1\end{cases}$$

It now holds that

$$\begin{pmatrix} 1 & 0 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 1 & -1 & 0\end{pmatrix}\begin{pmatrix} v(x_1)\\ v(x_2)\\ v(x_3)\\ v(x_4)\end{pmatrix}=\begin{pmatrix} 18\\ 13\\ 9\\ 9\\ 5\\ 4\end{pmatrix},$$

and hence $\succsim$ is indeed utility-difference representable by v. This is clearly not true for an arbitrary strictly increasing transformation $\widehat{v}$ of u [it is not, for example, when $\widehat{v} = \widehat{h} \circ u$, $\widehat{h}(z) := z^3$]. $\Diamond$

Example II

Suppose again that $X = \{x_1, x_2, x_3, x_4\}$ and let the binary relation $\succsim$ on $X \times X$ be such that

$$\begin{array}{ccccccccc} (x_1,x_4) & \succ & (x_2,x_4) & & & & & & \\ & \succ & (x_1,x_3) & \succ & (x_1,x_2) & \succ & (x_2,x_3) & \succ & (x_3,x_4) \\ & \succ & (x_1,x_1) & \sim & (x_2,x_2) & \sim & (x_3,x_3) & \sim & (x_4,x_4) \\ & \succ & (x_4,x_3) & \succ & (x_3,x_2) & \succ & (x_2,x_1) & \succ & (x_3,x_1) \\ & \succ & (x_4,x_2) & \succ & (x_4,x_1) & & & & \end{array}$$

This is a basic preference intensity relation that violates Concatenation: $(x_1, x_2) \succ (x_2, x_3)$ and $(x_2, x_3) \succ (x_3, x_4)$ but $(x_2, x_4) \succ (x_1, x_3)$. Nevertheless, it induces the same linearly ordered preference relation $\gtrapprox$ on X as in Example 1. Moreover, $\succsim$ is representable by the preference intensity function $s : X \times X \to \mathbb{R}$ defined by

$$\begin{aligned} s(x_1,x_4) &= 6 \\ s(x_2,x_4) &= 5 \\ s(x_1,x_3) &= 4 \\ s(x_1,x_2) &= 3 \\ s(x_2,x_3) &= 2 \\ s(x_3,x_4) &= 1 \\ s(x_1,x_1) = s(x_2,x_2) = s(x_3,x_3) = s(x_4,x_4) &= 0 \\ s(x_i,x_j) &= -s(x_j,x_i), \end{aligned}$$

which, not surprisingly, is *not* triangularly additive. $\Diamond$

Example III

Consider the following behavioural dataset $\mathcal{D} = \big(A_i, C(A_i), r_i\big)_{i=1}^{5}$:

$$\begin{array}{lll} A_1 = \{a,b,c\}, & C(A_1) = \{a\}, & r_1 = 17 \\ A_2 = \{a,c,d\}, & C(A_2) = \{a\}, & r_2 = 18 \\ A_3 = \{b,c,d\}, & C(A_3) = \{b\}, & r_3 = 15 \\ A_4 = \{b,c\}, & C(A_4) = \{b\}, & r_4 = 14 \\ A_5 = \{c,d\}, & C(A_5) = \{c\}, & r_5 = 6 \end{array}$$

It follows that

$$R_a = \{17, 18\}, \quad R_b = \{14, 15\}, \quad R_c = \{6\}.$$

Notice that $\mathcal{D}$ satisfies CMC:

$$\begin{array}{llllllll} a \ggcurly^R b, & \min R_a & = & 17 & > & 15 & = & \max R_b \\ b \ggcurly^R c, & \min R_b & = & 14 & > & 6 & = & \max R_c. \end{array}$$

Notice further that $\mathcal{D}$ also satisfies RDC:

$$\begin{array}{lllllll} \min\{R_a - R_c\} & = & 11 & > & 9 & = & \max\{R_b - R_c\} \\ \min\{R_b - R_c\} & = & 8 & > & 4 & = & \max\{R_a - R_b\} \end{array}$$

Therefore, it follows from Theorem 2 that $\mathcal{D}$ is rationalizable by a preference intensity function. To construct such a rationalizing s along the lines of the method that is developed in the proof of that result (and using the particular completion that is specified there), let $R_d := \{r_d\}$ for $r_d \in \mathbb{R}$ that satisfies

$$\min_{x \in \mathcal{D}_x}(r_x - r_d) > \max_{x,y \in \mathcal{D}_x}(r_x - r_y).$$

The case where $r_d := -7$ is such an example. Thus augmented, $\mathcal{D}$ has the following structure:

$$\begin{array}{lllllll} \min\{R_a - R_d\} & = & 24 & > & 22 & = & \max\{R_b - R_d\} \\ \min\{R_b - R_d\} & = & 21 & > & 13 & = & \max\{R_c - R_d\} \\ \min\{R_c - R_d\} & = & 13 & > & 12 & = & \max\{R_a - R_c\} \\ \min\{R_a - R_c\} & = & 11 & > & 9 & = & \max\{R_b - R_c\} \\ \min\{R_b - R_c\} & = & 8 & > & 4 & = & \max\{R_a - R_b\} \end{array}$$

The completed revealed preference intensity ordering $\gtrsim^R$ that is defined in this way is representable, for example, by the following $s : X \times X \to \mathbb{R}$:

$$\begin{array}{lll} s(a,d) & = & 6 \\ s(b,d) & = & 5 \\ s(c,d) & = & 4 \\ s(a,c) & = & 3 \\ s(b,c) & = & 2 \\ s(a,b) & = & 1 \\ s(x,y) & = & -s(y,x) \end{array}$$

$\Diamond$

Example IV

Consider the behavioural dataset $\mathcal{D}$ of Example III, modified so that $r_1 = r_2 = 18$ and $r_3 = r_4 = 15$. It now holds that $R_a = \{18\}$, $R_b = \{15\}$ and $R_c = \{6\}$. Therefore, CMC, RDC and RP are all satisfied, and it follows from Corollary 3 that $\mathcal{D}$ is cardinal-utility rationalizable. To construct such a rationalizing u, again let $R_d := \{-7\}$. Since it now holds that $\min\{R_x - R_y\} = \max\{R_x - R_y\} = R_x - R_y$ for all $x, y \in \{a, b, c, d\}$, define $s : X \times X \to \mathbb{R}$ by $s(x,y) := \min\{R_x - R_y\} = \max\{R_x - R_y\}$.

Then,

$$\begin{aligned}
s(a,d) &= 25\\
s(b,d) &= 22\\
s(c,d) &= 13\\
s(a,c) &= 12\\
s(b,c) &= 9\\
s(a,b) &= 3\\
s(x,y) &= -s(y,x)
\end{aligned}$$

This is a triangularly additive s that can be written $s(x,y) \equiv u(x) - u(y)$ for $u : X \to \mathbb{R}$ defined by $u(a) = 25$, $u(b) = 22$, $u(c) = 13$ and $u(d) = -7$. Moreover, u provides a cardinal-utility rationalization of $\mathcal{D}$ in the sense of (9). $\Diamond$

Appendix B: Proofs

Proof of Theorem 1.

It is straightforward to verify that if s is a preference intensity function that represents a binary relation $\succsim$ on $X \times X$, then $\succsim$ is a basic preference intensity ordering on X. To establish the converse implication we first make some auxiliary observations.

Claim 1. *If $\succsim$ satisfies Weak Order and Reversal, then $a \gtrapprox b$ and $c \gtrapprox d$ implies $(a,b) \succsim (d,c)$. Moreover, if either $a \gg b$ or $b \gg c$ is also true, then $(a,b) \succ (d,c)$.*

For the first part, let $a \gtrapprox b$, $c \gtrapprox d$ and suppose to the contrary that $(b,a) \succ (c,d)$. Transitivity and $(a,b) \succsim (b,a) \succ (c,d) \succsim (d,c)$ implies $(a,b) \succ (d,c)$. In view of Reversal, this is a contradiction. Moreover, in view of Completeness, $(b,a) \not\succ (d,c)$ implies $(d,c) \succsim (b,a)$, which, under Reversal, further implies $(a,b) \succsim (c,d)$, as required. For the second part, let $a \gg b$, $c \gtrapprox d$ and suppose to the contrary that $(a,b) \sim (d,c)$. Reversal implies $(c,d) \sim (b,a)$. Now noting that $(a,b) \succ (b,a)$ holds, Transitivity and $(d,c) \sim (a,b) \succ (b,a) \sim (c,d)$ together imply $(d,c) \succ (c,d)$, which contradicts $c \gtrapprox d$. The argument in the case where $a \gtrapprox b$ and $c \gg d$ is symmetric.

Claim 2. *If $\succsim$ satisfies Completeness, Separability and Reversal, then $(c,a) \succsim (c,b)$ implies $(d,a) \succsim (d,b)$, and $(c,a) \succ (c,b)$ implies $(d,a) \succ (d,b)$.*

For the first part, suppose to the contrary that $(c,a) \succsim (c,b)$ and $(d,a) \not\succsim (d,b)$. Completeness implies $(d,b) \succ (d,a)$. Under Reversal, $(d,b) \succ (d,a)$ implies $(a,d) \succ (b,d)$, and $(c,a) \succsim (c,b)$ implies $(b,c) \succsim (a,c)$. But Separability and $(b,c) \succsim (a,c)$ together imply $(b,d) \succsim (a,d)$, which contradicts $(a,d) \succ (b,d)$. For the second part, suppose to the contrary that $(c,a) \succ (c,b)$ and $(d,a) \sim (d,b)$. Reversal then implies $(b,c) \succ (a,c)$ and $(b,d) \sim (a,d)$. The latter implies $(a,d) \succsim (b,d)$, which, together with Separability, contradicts $(b,c) \succ (a,c)$.

Claim 3. *If $\succsim$ satisfies Transitivity, Separability and Reversal, then $(a,c) \succsim (b,c)$ implies $a \gtrapprox b$, and $(a,c) \succ (b,c)$ implies $a \gg b$.*

For the first part, note that Separability and $(a,c) \succsim (b,c)$ implies $(a,b) \succsim (b,b)$, which, under Reversal, further implies $(b,b) \succsim (b,a)$. It now follows from Transitivity and $(a,b) \succsim (b,b) \succsim (b,a)$

that $(a,b) \succsim (b,a)$, or $a \gtrapprox b$. For the second part, suppose to the contrary that $(a,c) \succ (b,c)$ and $(a,b) \sim (b,a)$. Reversal and $(a,c) \succ (b,c)$ together imply $(c,b) \succ (c,a)$. It now follows from this and Claim 2 that $(b,b) \succ (b,a)$, which, under Reversal, further implies $(a,b) \succ (b,a)$, a contradiction.

Claim 4. *If $\succsim$ is a basic preference intensity relation on X, then $\gtrapprox$ is a weak order on X.*

Completeness of $\gtrapprox$ immediately follows from its definition and the assumed completeness of $\succsim$. For transitivity, suppose $a \gtrapprox b$ and $b \gtrapprox c$, and assume to the contrary that $c \ggcurly a$. We have $(a,b) \succsim (b,a)$, $(b,c) \succsim (c,b)$ and $(c,a) \succ (a,c)$. In view of Claim 1, $(a,b) \succsim (b,a)$ and $(c,a) \succ (a,c)$ implies $(a,b) \succ (a,c)$. Claim 2 now implies $(b,b) \succ (b,c)$, which, under Reversal, further implies $c \ggcurly b$, a contradiction.

Claim 5. *If $\succsim$ is a basic preference intensity relation on X, then $(a,b) \succsim (b,a)$ and $(b,c) \succsim (c,b)$ implies $(a,c) \succsim (a,b)$ and $(a,c) \succsim (b,c)$.*

Indeed, let $(a,b) \succsim (b,a)$ and $(b,c) \succsim (c,b)$, and suppose to the contrary that $(a,b) \succ (a,c)$. It is implied by Claim 2 that $(c,b) \succ (c,c)$. Transitivity and $(b,c) \succsim (c,b) \succ (c,c)$ further imply $(b,c) \succ (c,c)$. But since $(a,b) \succ (a,c)$ holds, Claim 2 also implies that $(c,b) \succ (c,c)$, which is a contradiction.

Moving on now, note that, since X is finite and $\gtrapprox$ is a weak order on X (Claim 4), there exist k $\approx$-equivalence classes $[x_i]$ which, with a slight abuse of notation, can be strictly ordered as

$$[x_1] \ggcurly \ldots \ggcurly [x_k].$$

The above ordering will be held fixed throughout the proof. In particular, it will be understood that, for any $i \leqslant k$, $x, y \in [x_i] \Leftrightarrow x \approx y$ and also that, for any $i < j$, $x \in [x_i]$ and $y \in [x_j] \Leftrightarrow x \ggcurly y$.

Let the $\approx$-quotient set of X be defined by $X_{\approx} \equiv \mathcal{X} := \{[x_1], \ldots, [x_k]\}$. Let also

$$A \;:=\; \{[x_i] \times [x_j] \in \mathcal{X} \times \mathcal{X} : i < j\}$$

and

$$Q_{\succ}(x_i, x_j) \;:=\; \{[x_h] \times [x_s] \in A : (x_i, x_j) \succ (x_h, x_s)\}$$

That is, $[x_h] \times [x_s] \in Q_{\succ}(x_i, x_j)$ if and only if, for all $(x_h, x_s) \in [x_h] \times [x_s]$, $x_h \ggcurly x_s$ and $(x_i, x_j) \succ (x_h, x_s)$. Notice that $Q_{\succ}(x_i, x_j) \neq \varnothing$ implies $i < j$ but the converse is not true in general.

Now define the function $s : X \times X \to \mathbb{R}$ by

$$s(x_i, x_j) = \begin{cases} 1 + |Q_{\succ}(x_i, x_j)|, & \text{if } i < j \\ 0, & \text{if } i = j \\ -s(x_j, x_i), & \text{if } i > j \end{cases}$$

Note that this s is well-defined in $X \times X$ since $(x_i, x_j) \in X \times X$ if and only if $(x_i, x_j) \in [x_i] \times [x_j]$ for some $[x_i],[x_j] \in \mathcal{X}$ where, clearly, exactly one of $i < j$, $i = j$ and $i > j$ is true. Moreover, s satisfies (3b) by construction. We will show that s also satisfies (3a), and it will then follow from Claim 5 that s obeys (3c) as well.

Notice first that it follows from the definitions of s and $Q_{\succ}(\cdot)$, and also from $[x_1] \ggcurly \ldots \ggcurly [x_k]$, that $s(x_i,x_j) > 0 \Leftrightarrow x_i \ggcurly x_j$ and $s(x_i,x_j) = 0 \Leftrightarrow x_i \approx x_j$. Now suppose $(x_j,x_l) \succsim (x_m,x_n)$ and assume $j \leqslant l$. It holds that $[x_m] \times [x_n] \in Q_{\succ}(x_j,x_l)$ or $Q_{\succ}(x_j,x_l) = \varnothing$. Given the definitions of s and $Q_{\succ}(\cdot)$, the first case implies $s(x_j,x_l) > s(x_m,x_n)$ because $(x_j,x_l) \succ (x_m,x_n)$ and therefore $Q_{\succ}(x_j,x_l) \supset Q_{\succ}(x_m,x_n)$ since $\succsim$ is a weak order on $X \times X$. The second case, $Q_{\succ}(x_j,x_l) = \varnothing$, implies $s(x_j,x_l) = 1$. Moreover, if $Q_{\succ}(x_j,x_l) = \varnothing$ and $m \leqslant n$, then $(x_j,x_l) \succsim (x_m,x_n)$ implies $(x_j,x_l) \sim (x_m,x_n)$, which further implies $s(x_m,x_n) = 1 = s(x_j,x_l)$. On the other hand, $Q_{\succ}(x_j,x_l) = \varnothing$ and $m > n$ implies $s(x_m,x_n) < 0 < s(x_j,x_l) = 1$. Assume now that $j > l$. In view of Claim 1, this implies $m > n$. Reversal now implies $(x_n,x_m) \succsim (x_l,x_j)$. Applying the above argument to this case establishes that $s(x_n,x_m) \geqslant s(x_l,x_j)$ and, given that $s(a,b) = -s(b,a)$ for all $a,b \in X$ is true by construction, $s(x_j,x_l) \geqslant s(x_m,x_n)$. Thus, for all $x_j,x_l,x_m,x_n \in X$, $(x_j,x_l) \succsim (x_m,x_n)$ implies $s(x_j,x_l) \geqslant s(x_m,x_n)$. Conversely, suppose $s(x_j,x_l) \geqslant s(x_m,x_n)$. Assume to the contrary that $(x_j,x_l) \not\succsim (x_m,x_n)$. Since $\succsim$ is complete, this implies $(x_m,x_n) \succ (x_j,x_l)$. It now follows from the above arguments that $s(x_m,x_n) > s(x_j,x_l)$, a contradiction. Therefore, s represents $\succsim$ as in (3).

To establish the uniqueness property, let s be a preference function that represents $\succsim$ and let t be an odd and strictly increasing transformation of s. Since $s(a,b) = -s(b,a)$ and $t(a,b) = f(s(a,b))$ for some function $f : \mathbb{R} \to \mathbb{R}$ that is odd in $s(X \times X)$, we have

$$t(a,b) = f(s(a,b)) = -f(-s(a,b)) = -f(s(b,a)) = -t(b,a).$$

Now suppose $(a,b) \succsim (c,d)$. This is equivalent to $s(a,b) \geqslant s(c,d)$. Since t is a strictly increasing transformation of s, it follows that $t(a,b) \geqslant t(c,d)$ too. Conversely, suppose $\succsim$ is represented by two distinct preference intensity functions s and t. Let $t := f \circ s$ for some function $f : \mathbb{R} \to \mathbb{R}$. Suppose $f(-z) \neq -f(z)$ for some $z \in s(X \times X)$. Let $z = s(a,b)$. Since $s(a,b) = -s(b,a)$ and $t(a,b) = f(s(a,b))$, by assumption, it follows that $t(b,a) = f(s(b,a)) = f(-s(a,b)) \neq -f(s(a,b)) = -t(a,b)$, which contradicts the assumption that t represents $\succsim$. Therefore, f is odd in $s(X \times X)$. Now suppose $f(z) \leqslant f(z')$ for some $z,z' \in s(X \times X)$ such that $z > z'$. Suppose $z = s(a,b)$ and $z' = s(c,d)$. By assumption, $(a,b) \succ (c,d)$. Since $f(z) = f(s(a,b)) = t(a,b) \leqslant t(c,d) = f(s(c,d)) = f(z')$, this again contradicts the assumption that t represents $\succsim$. Therefore, f is strictly increasing in $s(X \times X)$. ■

Proof of Corollary 2.

1 ⇔ 2. See Theorem 3.2 in Scott (1964).

2 ⇒ 3. Defining $s : X \times X \to \mathbb{R}$ by $s(x,y) := u(x) - u(y)$ trivially establishes the claim.

3 ⇒ 2. Let z be an arbitrary element of X. Suppose $\succsim$ is represented by the triangularly additive

preference intensity function $s : X \times X \to \mathbb{R}$. We have

$$\begin{aligned} a \gtrapprox b &\iff s(a,b) \geqslant s(b,a) \\ &\iff s(a,b) \geqslant 0 \\ &\iff s(a,b) + s(b,z) \geqslant s(b,z) \\ &\iff s(a,z) \geqslant s(b,z) \end{aligned} \tag{10}$$

where the last step makes use of the fact that s is triangularly additive. Now define the function $u : X \to \mathbb{R}$ by

$$u(a) := s(a,z). \tag{11}$$

It follows from (10) and (11) that

$$\begin{aligned} a \gtrapprox b &\iff s(a,z) \geqslant s(b,z) \\ &\iff u(a) \geqslant u(b). \end{aligned}$$

Thus,

$$\begin{aligned} (a,b) \gtrsim (c,d) &\iff s(a,b) \geqslant s(c,d) \\ &\iff s(a,z) + s(z,b) \geqslant s(c,z) + s(z,d) \\ &\iff s(a,z) - s(b,z) \geqslant s(c,z) - s(d,z) \\ &\iff u(a) - u(b) \geqslant u(c) - u(d). \end{aligned}$$

■

Proof of Theorem 2.

Showing that CMC and RDC are implied by (6) is straightforward and omitted. To establish the converse implication, start by defining

$$R_z := \{r\} \ \forall \ z \in X \backslash \mathcal{D}_x, \text{ where } r \in \mathbb{R} \text{ is such that } \min_{x \in \mathcal{D}_x}(r_x - r) > \max_{x,y \in \mathcal{D}_x}(r_x - r_y). \tag{12}$$

Next, define the binary relation $\gtrsim$ on $X \times X$ by

$$(a,b) \gtrsim (c,d) \iff \begin{cases} (c,d) = (a,b) \\ \\ \text{or} \\ \\ \min\{R_a - R_b\} \geqslant \max\{R_c - R_d\} \end{cases} \tag{13}$$

This relation is reflexive by definition. Moreover, (12) and (13) together imply

$$z, z' \in X\backslash\mathcal{D}_x,\ y \in X,\ x \in \mathcal{D}_x \implies \begin{cases} (z, z') \sim (y, y) \\ \& \\ (x, z) \succ (z, x) \end{cases} \tag{14}$$

It now follows from (12), (13), (14) and RDC that $(a, b) \gtrsim (c, d)$ or $(c, d) \gtrsim (a, b)$ for all $a, b, c, d \in X$. Hence, $\gtrsim$ is complete. Moreover, (13) and (14) imply that, for all $a, b \in X$,

$$\min\{R_a - R_b\} \equiv (-1) \cdot \max\{R_b - R_a\}.$$

Therefore, $\gtrsim$ also satisfies Reversal.

We will now show that $\gtrsim$ is transitive in $X \times X$. It follows directly from (13) that $(a, b) \gtrsim (c, d) \gtrsim (e, f)$ implies $(a, b) \gtrsim (e, f)$ whenever $a, b, c, d, e, f \in \mathcal{D}_x$. We will verify that transitivity holds in general by considering: a) elements of $X\backslash\mathcal{D}_x \times X\backslash\mathcal{D}_x$ only; b) elements of $\mathcal{D}_x \times X\backslash\mathcal{D}_x$ only; c) elements of $X\backslash\mathcal{D}_x \times \mathcal{D}_x$ only; d) elements of $X\backslash\mathcal{D}_x \times X\backslash\mathcal{D}_x$, $\mathcal{D}_x \times X\backslash\mathcal{D}_x$ and $X\backslash\mathcal{D}_x \times \mathcal{D}_x$.

Consider case a) first. If $(a, b), (c, d), (e, f) \in X\backslash\mathcal{D}_x \times X\backslash\mathcal{D}_x$, then (14) directly implies $(a, b) \sim (c, d) \sim (e, f)$ and $(a, b) \sim (e, f)$. Hence, transitivity holds in this case. For b), suppose $(a, b) \gtrsim (c, d) \gtrsim (e, f)$, with these pairs being elements of $\mathcal{D}_x \times X\backslash\mathcal{D}_x$. In view of (12), (13) and (14), $(a, b) \gtrsim (c, d)$ and $(c, d) \gtrsim (e, f)$ are equivalent to $\min\{R_a - r\} \geqslant \max\{R_c - r\}$ and $\min\{R_c - r\} \geqslant \max\{R_e - r\}$, respectively. These imply $\min R_a \geqslant \max R_c$ and $\min R_c \geqslant \max R_e$, which in turn imply $\min R_a \geqslant \max R_e$, hence $\min\{R_a - r\} \geqslant \max\{R_e - r\}$. The latter is equivalent to $(a, b) \gtrsim (e, f)$. Transitivity is therefore satisfied here as well. The argument for case c) is symmetric.

For case d), suppose $(a, b) \gtrsim (c, d) \gtrsim (e, f)$ without restricting all three pairs to belong to the same particular subset of $X \times X$ from those considered above. To this end, suppose first that $(a, b) \in X\backslash\mathcal{D}_x \times X\backslash\mathcal{D}_x$. It readily follows then that $(c, d) \in X\backslash\mathcal{D}_x \times X\backslash\mathcal{D}_x$ or $(c, d) \in X\backslash\mathcal{D}_x \times \mathcal{D}_x$. Suppose $(c, d) \in X\backslash\mathcal{D}_x \times X\backslash\mathcal{D}_x$. It now follows from $(c, d) \gtrsim (e, f)$ that $(e, f) \in X\backslash\mathcal{D}_x \times X\backslash\mathcal{D}_x$ or $(e, f) \in X\backslash\mathcal{D}_x \times \mathcal{D}_x$. In both cases, adapting the above argument establishes that $(a, b) \gtrsim (e, f)$. Now suppose $(c, d) \in X\backslash\mathcal{D}_x \times \mathcal{D}_x$. It now holds that $(e, f) \in X\backslash\mathcal{D}_x \times \mathcal{D}_x$ too, and $(a, b) \succ (e, f)$. Consider now the case where $(a, b) \in X \times \mathcal{D}_x$. If $(c, d), (e, f) \in X \times \mathcal{D}_x$, then we are in case b) above. Therefore, suppose $(c, d) \in X \times \mathcal{D}_x$ and either $(e, f) \in X\backslash\mathcal{D}_x \times X\backslash\mathcal{D}_x$ or $(e, f) \in X\backslash\mathcal{D}_x \times \mathcal{D}_x$. In light of the above, it is easy to verify that $(a, b) \gtrsim (e, f)$ (similarly in the case where $(c, d) \in X\backslash\mathcal{D}_x \times X\backslash\mathcal{D}_x$ or $(c, d) \in X\backslash\mathcal{D}_x \times \mathcal{D}_x$). Finally, consider the case where $(a, b) \in X\backslash\mathcal{D}_x \times \mathcal{D}_x$. It readily follows from the above that $(c, d), (e, f) \in X\backslash\mathcal{D}_x \times \mathcal{D}_x$ as well, which is covered by case c).

To show that $\gtrsim$ also satisfies Separability on $X \times X$, let $(a, c) \gtrsim (b, c)$ and assume to the contrary that $(b, d) \succ (a, d)$ for some $d \neq c$, where $a, b, c, d \in X$. Given (12), (13) and (14), and recalling (7) which also applies for the asymmetric relation $\succ$ on $X \times X$ that is induced by (13), these postulates imply

$$\begin{aligned} \min\{R_a - R_c\} &\geqslant \max\{R_b - R_c\} \\ \min\{R_b - R_d\} &\geqslant \max\{R_a - R_d\} \\ \max\{R_b - R_d\} &> \max\{R_a - R_d\}. \end{aligned}$$

The first two inequalities imply

$$\begin{array}{rcccl} \min R_a - \max R_b & \geqslant & \max R_c - \min R_c & \geqslant & 0 \\ \min R_b - \max R_a & \geqslant & \max R_d - \min R_d & \geqslant & 0 \end{array}$$

and hence

$$\min R_a - \max R_b \quad = \quad \min R_b - \max R_a,$$

which is true if and only if

$$R_a = R_b = \{\overline{r}\},$$

for some $\overline{r} \in \mathbb{R}$. However, since $\max\{R_b - R_d\} > \max\{R_a - R_d\}$ is also true, and this is equivalent to

$$\max R_b - \max R_a > 0,$$

we arrive at a contradiction. Therefore, it has been shown that $\succsim$ is a Weak Order on $X \times X$ that also satisfies Reversal and Separability, hence that it is a basic preference intensity relation on X.

Now, towards defining the relation $\succapprox$ on X that is induced by $\succsim$, observe that

$$\begin{array}{rcl} (a,b) \succsim (b,a) & \Longleftrightarrow & \min\{R_a - R_b\} \geqslant 0 \geqslant \max\{R_b - R_a\} \\ & \Longleftrightarrow & \min R_a \geqslant \max R_b \\ & \Longleftrightarrow & a \succapprox b \end{array}$$

Notice that, in light of Claim 4 in the proof of Theorem 1, $\succapprox$ is a weak order on X. Next, recall the binary relations $\succapprox^R$, $\gg^R$ and $\approx^R$ of direct revealed weak preference, strict preference and indifference on X that are defined by

$$\begin{array}{rcl} x \succapprox^R y & \Longleftrightarrow & x \in C(A),\ y \in A \quad \text{for some } A \in \mathcal{D}_A \\ x \gg^R y & \Longleftrightarrow & x \succapprox^R y \ \text{ and } \ y \not\succapprox^R x \\ & \Longleftrightarrow & x \in C(A),\ y \in A \backslash C(A) \quad \text{for some } A \in \mathcal{D}_A \\ x \approx^R y & \Longleftrightarrow & x \succapprox^R y \ \text{ and } \ y \succapprox^R x \end{array}$$

It follows from CMC that $\gg^R$ is acyclic and, in addition, that

$$\begin{array}{rcl} x \succapprox^R y & \Longrightarrow & x \succapprox y \\ x \gg^R y & \Longrightarrow & x \gg y \\ x \approx^R y & \Longrightarrow & x \approx y. \end{array}$$

Therefore, $\succapprox$ is a weak-order completion of $\succapprox^R$ on X.

Since $\succsim$ is a basic preference intensity relation on the finite set X, it follows from Theorem 1 that it is representable by a preference intensity function $s : X \times X \to \mathbb{R}$. Moreover, since the weak order $\succapprox$ that is induced by $\succsim$ is a completion of the direct weak revealed preference relation $\succapprox^R$ on X, s clearly satisfies (6a). Finally, (12), (13) and RDC ensure that s also satisfies (6b). Therefore, s rationalizes $\mathcal{D}$. ■