Ordinal Intensity-Efficient Allocations

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Abstract

We study the assignment problem in situations where, in addition to having ordinal preferences, agents also have *ordinal intensities*: they can make simple and internally consistent comparisons such as "I prefer a to b more than I prefer c to d" without necessarily being able to quantify them. In this new informational social-choice environment we first introduce a rank-based criterion that enables interpersonal comparability of such ordinal intensities. Building on this criterion, we define an allocation to be *"intensity-efficient"* if it is Pareto efficient with respect to the preferences induced by the agents' intensities and also such that, when another allocation assigns the same pairs of items to the same pairs of agents but in a "flipped" way, the former allocation assigns the commonly preferred item in every such pair to the agent who prefers it more. We present some first results on the (non-)existence of such allocations without imposing restrictions on preferences or intensities other than strictness.

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The problem I have with utilitarianism is not that it is excessively rational, but that the epistemological foundations are weak. My problem is: What are those objects we are adding up? I have no objection to adding them up if there's something to add.

Kenneth J. Arrow (1987)[1]

Suppose I am left with a ticket to a Mozart concert I am unable to attend and decide to give it to one of my closest friends. Which friend should I actually give it to? One thing I will surely consider in deciding this is which friend of mine would enjoy the concert most.

John C. Harsanyi (1990)

1 Introduction

In this paper we propose and study a refinement of Pareto efficiency in the novel analytical environment where agents can express their ordinal preferences as well as their *ordinal preference intensity comparisons*. In particular, similar to the way in which preferences are elicited in various matching markets, we assume that information about agents' intensities can be obtained by asking them to respond to simple questions such as *"Do you prefer a to b more than you prefer c to d?"*, and that these responses are internally consistent in a way that we make precise. Crucially, unlike existing work on intensity-inclusive social choice or allocation problems, we do not assume that those comparisons are necessarily quantifiable/(pseudo-)cardinalizable, although such more refined perceptions are indeed encompassed as a special case in the class of intensity relations that we consider.

Operating within this new informational setting for social choice, we are interested in a distributively just assignment of n indivisible items to n agents when monetary transfers are infeasible and agents are not assumed to have preferences over lotteries over allocations. Thus, as is often the case in practice, we rule out the agents' willingness to pay for the different items from being a potential source of information about their generally differing preference intensities. In addition, we rule out the possibility of the agents' attitudes towards risk acting as confounds of their preference intensities (Arrow, 2012/1951; Ellingsen, 1994; Schoemaker, 1982; Sen, 2017/1970).

This non-cardinal and non-utilitarian framework where preference and preference-intensity information is nevertheless still available to the social planner/matching-platform designer naturally raises the question of how this information might be used to arrive at some normatively appealing refinement of Pareto efficiency that would also reflect differences in the agents' preference intensities. Similar to utilitarian or other cardinal-utility notions of efficiency, such a refinement requires that one be able to make some kind of interpersonal comparisons (Baccelli, 2023; Echenique, Immorlica, and Vazirani, 2023; Fleurbaey and Hammond, 2004). Unlike those notions, however, in our framework interpersonal comparisons cannot be based on the agents' utilities and must rely on the information contained in the above ordinal intensity rankings.

To make progress against the background of this analytical challenge we assume that such comparisons can be made when all agents' preferences and intensities are strict by contrasting the rank-order position of pairs of alternatives (a, b) in the different agents' intensity orderings. In particular, when both agents i and j prefer a to b but the pair (a, b) lies higher in i's intensity ranking than in j's, then we assume that i prefers it *more*. Building on these ordinal and unweighted interpersonal intensity comparisons, we then define an allocation x as *intensity-efficient* if it is Pareto efficient and also such that, whenever

[1] Quoted in Ellingsen (1994)

another allocation y assigns the same pairs of objects to the same pairs of agents but in a "flipped" way, i.e. when $(x_i, x_j) = (y_j, y_i) = (a, b)$ for agents i, j and alternatives a, b, then x assigns the commonly preferred alternative in each such pair to the agent who prefers it more.

We show that, without any further restrictions, an intensity-efficient allocation exists for all strict intensity profiles when there are three agents and alternatives. With five or more, however, the existence of intensity-efficient allocations is not guaranteed because the underlying dominance relation may be cyclic if additional restrictions are not imposed.

2 Setup and Decision-Theoretic Foundations

There is a finite set X of $n \geq 3$ indivisible objects to be allocated to n agents. Agent $i \leq n$ is assumed to have a *preference intensity relation* $\dot{\succsim}_i$ –the primitive of our analysis– on X. In line with the literature on extensive measurement (Krantz, Luce, Suppes, and Tversky, 1971; Pfanzagl, 1971; Roberts, 1979), this is taken to be a binary relation over pairs in $X \times X$ or, equivalently, a quaternary relation over elements of X. Correspondingly, we interpret $(a, b) \dot{\succsim}_i (c, d)$ as capturing the situation where a is better (worse) than b more (less) than c is relative to d. The relations $\dot{\succ}$ and $\dot{\sim}$ emerge as the asymmetric and symmetric parts of $\dot{\succsim}$ and capture the strict intensity and intensity-equivalence relations, respectively. Also in line with that literature (and with common sense), we identify the *preference relation* that is induced by $\dot{\succsim}_i$ with $a \succsim_i b \Leftrightarrow (a, b) \dot{\succsim}_i (b, a)$.

Throughout this paper we focus on intensity relations which, in addition to three core properties that we discuss below, satisfy the following:

Strictness
For all $a, b, c, d \in X$, $(a, b) \dot{\sim} (c, d)$ iff $(a, b) = (c, d)$ or $a = b$ and $c = d$.

This condition implies that no agent perceives an intensity-equivalence between distinct pairs of distinct alternatives. It is therefore analogous to the familiar strictness condition of preferences that rules out indifferences (and which, in fact, it implies).

We assume that $\dot{\succsim}_i$ is representable by some ordinal and strict *preference intensity function*, i.e. a mapping $s_i : X \times X \to \mathbb{R}$ with the following properties:

$$(a, b) \dot{\succ}_i (c, d) \iff s_i(a, b) > s_i(c, d) \quad (1a)$$
$$s_i(a, b) = -s_i(b, a) \quad (1b)$$
$$\min\{s_i(a, b),\ s_i(b, c)\} > 0 \implies s_i(a, c) > \max\{s_i(a, b), s_i(b, c)\} \quad (1c)$$

That is: (i) a higher and positive from left to right preference-intensity difference between the two alternatives in a pair is associated with a higher value of s_i at that pair; and (ii) intensity differences are monotonically increasing in the underlying preference ordering. The class of intensity relations that admit such a representation in the more general case where Strictness is not imposed is characterized by the following three axioms (Gerasimou, 2021): (i) Weak Order; (ii) $(a, b) \dot{\succsim}_i (c, d) \Leftrightarrow (d, c) \dot{\succsim}_i (b, a)$ (Reversal); (iii) $(a, c) \dot{\succsim}_i (b, c) \Leftrightarrow (a, b) \dot{\succsim}_i (b, a)$ (Translocation Consistency).[2]

[2]When Strictness is assumed, (1a)–(1c) is characterizable with a one-way implication sign instead of an equivalence sign in the statement of Translocation Consistency.

Importantly, preference intensity functions are unique up to a transformation by some $f : \mathbb{R} \to \mathbb{R}$ that is odd $[f(-z) = -f(z)]$ and strictly increasing in the relevant range. The oddness property here ensures that the *skew-symmetry* condition (1b) is preserved.[3] This condition in turn is without loss of generality and simplifies the notation, as it implies that $a \succsim_i b \Leftrightarrow s_i(a, b) \geq 0$. Because this convenient normalization of s is without loss of generality (see Theorem 1 in Gerasimou, 2021), the combination (1a)–(1c) essentially constitutes an *ordinal* representation.

An special case of this model that is of particular interest emerges when $\dot{\succsim}_i$ admits a *utility-difference representation*. In this case, there exists some function $u_i : X \to \mathbb{R}_+$ such that

$$(a, b) \dot{\succsim}_i (c, d) \iff u_i(a) - u_i(b) \geq u_i(c) - u_i(d) \tag{2}$$

When such a representation is possible, s_i in (1a)–(1c) can be defined by $s_i(a, b) \equiv u_i(a) - u_i(b)$. This decomposition can be made in turn if and only if the *lateral consistency* condition (1c), which ties together harmoniously the intensity relation and the preference relation induced by it and ensures transitivity of the latter, is strengthened to *additivity*, which requires $s_i(a, c) = s_i(a, b) + s_i(b, c)$ for all $a, b, c \in X$.[4] Interestingly, when such a utility-difference representation exists, it is unique neither up to a positive affine transformation nor up to an arbitrary strictly increasing transformation. Instead, it is invariant to certain non-cardinal and strictly increasing transformations that are $\dot{\succsim}_i$-dependent and cannot be easily summarized.[5] An implication of this fact is that these utility indices, even if normalized so that they have the same range for all agents, cannot be interpreted as precise units of measurement. These *pseudo-cardinal* utility representations were first characterized in Scott (1964) by means of Completeness and a hard-to-interpret Cancellation axiom.

Table 1 enumerates the number of distinct utility-difference and preference-intensity representable intensity relations in small finite sets when Strictness is assumed and juxtaposes them to the corresponding number of strict preference relations.

Table 1: Distinct ordinal utility, pseudo-cardinal utility and ordinal preference-intensity functions in small domains that are possible under Strictness.

n	Strict ordinal utility representations	Strict pseudo-cardinal utility-difference representations	Strict ordinal preference intensity representations
3	6	12	12
4	24	240	384
5	120	13,680	92,160

Source for 3rd & 4th columns: MiniZinc computations with the Gecode solver.

This table clarifies that, with 3 alternatives in the domain, the two models coincide under Strictness. It also clarifies, however, that the ordinal model can account for considerably

[3]To the best of our knowledge, skew-symmetric functions were apparently introduced in economics with Samuelson (1938) and his bivariate-function reformulation of neoclassical cardinal utility representations of preference intensities. Such functions later gained wider prominence in the literature of intransitive preference representations.

[4]The latter fact is well-known and due to the solution to Sincov's functional equation (Aczél, 1966).

[5]Having said that, we note that Baccelli (2024) recently showed that any ordinal utility representation of a preference relation induces an incomplete set of utility-difference comparisons that are, in fact, preserved by *arbitrary* strictly increasing transformations.

more preference intensity comparisons than the pseudo-cardinal one when there are more than 3 alternatives, with explanatory gains that are increasing in that number (60% and 573% when $n = 3$ and $n = 4$, respectively).

Naturally, one may also be interested in the further refined version of (2) where the utility index is *cardinal* in the sense that it is unique up to a positive affine transformation. On finite choice sets, however, such representations are possible only in very limiting environments. In particular, only when the intensity difference between any two consecutively-ranked alternatives is the same are cardinal utility differences known to be possible in finite environments (Krantz, Luce, Suppes, and Tversky, 1971, Theorem 5, p. 168). Importantly, however, this very special but also well-defined class of intensity relations is implicitly or explicitly invoked in cardinal interpretations of the classic *Borda rule* (Borda, 1781) of preference aggregation, most recently (and directly) in Maskin (2024). While we do not touch the problem of preference aggregation in this paper, we note that the class of intensity relations and ordinal representations thereof that we consider here can be used towards generalizing the Borda rule in ways that allow for "non-linear" intensities as well. The next section is more illuminating in this regard.

3 Interpersonally Comparable Ordinal Intensities

Under Strictness we can assume without loss of generality that each $\dot{\succsim}_i$ is represented by a *canonical* s_i in the sense that

$$s_i(X \times X) = \{-k, \ldots, -2, -1, 0, 1, 2, \ldots, k\},$$

where $k \equiv \binom{n}{2} - n$ is the number of distinct pairs of distinct alternatives in X. That is, every agent's intensity function is *onto* the same set of consecutive integers that is symmetric around zero.

Focusing on pairs (a, b) where a is preferred to b, this canonical normalization implies that the value of the agents' canonical intensity functions at such a pair reflects the rank/position of that pair in the respective agents' strict intensity rankings. It allows in turn for a novel kind of meaningful *ordinal* interpersonal comparisons of preference intensities to be assumed, without also requiring interpersonally comparable *utilities* –cardinal, pseudo-cardinal, or otherwise. At the same time, however, these interpersonal comparisons of canonically represented preference intensities are analogous to those in:

1. *Relative Utilitarianism*, where each $\succ_i$ is defined over lotteries over alternatives in X and represented by the von Neumann and Morgenstern (1947) *cardinal* u_i where $u_i(X) := [0, 1]$ (Harsanyi, 1955; Dhillon and Mertens, 1999);

2. *Borda-count aggregation*, where each $\succ_i$ is defined over alternatives in X and represented by the *ordinal* u_i with $u_i(a) := |\{a' \in X : a \succ_i a'\}|$, $u_i(X) \equiv \{0, 1, \ldots, |X| - 1\}$ (Borda, 1781).

Proceeding with the task at hand, an *intensity profile* is an n-tuple $(\dot{\succsim}_1, \ldots, \dot{\succsim}_n)$ of Strictness-complying preference intensity relations, one for each agent $i \leq n$. Every such profile corresponds to a unique *canonical intensity function profile* $s = (s_1, \ldots, s_n)$.

Comparability of Ordinal Intensities
Given a strict preference intensity profile $\dot{\succsim} = (\dot{\succsim}_1, \ldots, \dot{\succsim}_n)$ *that is canonically represented*

by $s = (s_1, \ldots, s_n)$, the statement

$$s_i(a, b) > s_j(a, b) > 0$$

is assumed to imply that agent i prefers a to b more *than j does.*

Towards motivating this new assumption, which is central in the paper, let us recall that our underlying model of preference intensities at the level of the individual decision maker does not assume that the intensity comparisons of any agent can be quantified with any precision beyond the level of an ordinal ranking. Then, under the maintained assumption of such simple/non-necessarily-quantifiable intensities, the theoretical question is whether the different agents' intensity-difference rankings should be treated *equally* by the social planner/matching-platform designer or not. In particular, is it the case that, in the absence of information regarding the degree to which agents i and j would suffer if they received b instead of a, the planner should declare that i prefers a to b *more* than j if all that the planner knows is that the former intensity difference lies higher in i's ranking than the latter does in j's? Since the agents' separate intensity orderings convey all the available welfare-relevant information, treating them in any way other than equal would call for a justification that appears elusive. Our assumption therefore might be thought of as a reasonable starting point for interpersonal comparisons in such an environment.

4 Intensity-Efficient Allocations

Building on the comparability assumption of the previous section, we can now introduce the following novel notions of dominance and efficiency.

Definition 1
Let $\dot{\succsim} = (\dot{\succsim}_1, \ldots, \dot{\succsim}_n)$ be a strict intensity profile and $\succ = (\succ_1, \ldots, \succ_n)$ be its induced strict-preference profile. Let s_i be the canonical representation of $\dot{\succsim}_i$. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be Pareto allocations with respect to $\succ$. Then, x intensity-dominates *y if for every pair of agents (i, j) such that*

$$(x_i, x_j) \quad = \quad (y_j, y_i)$$

it holds that

$$s_i(x_i, x_j) \quad \geq \quad s_j(y_j, y_i),$$

and there is at least one pair (i, j) where this inequality is strict.

Definition 2
A Pareto efficient allocation is intensity-efficient *if it is not intensity-dominated.*

Definition 3
Two intensity-efficient allocations x and y are equivalent *if $s_i(x_i, x_j) = s_j(y_j, y_i)$ for all pairs of agents (i, j) such that $(x_i, x_j) = (y_j, y_i)$ while $x_k = y_k$ for every other agent k.*

More descriptively, a Pareto efficient allocation x intensity-dominates another such allocation y if, in every pair of agents that is "flipped" by x and y in the sense that both allocations assign the same two alternatives a and b to the two agents in that pair but do so in opposite ways, the agent receiving a (which, under the postulated Pareto efficiency, is the mutually preferred one) under x prefers it to b *more* than the agent receiving it under y. Therefore, if allocations x and y are Pareto efficient and x intensity-dominates y, then the interpersonal preference trade-offs in all pairs of agents that receive the same two alternatives under x and y but in reverse order are always resolved by x in favour of the agent in the pair who prefers the relevant alternative more. Thus, the concept of intensity efficiency conforms with intuitive principles of distributive justice. Moreover, it appears to be the first refinement of Pareto efficiency that is operational in an environment where neither the agents' utilities are required to be inter- and intra-personally comparable nor monetary transfers between agents are assumed to be feasible.

Theorem 1
An intensity-efficient allocation exists for every intensity profile when $n = 3$.

Proof of Theorem 1.
From Table 1, there are $12^3 = 1,728$ unique preference intensity profiles under Strictness. The argument proceeds by considering the possible ways in which an arbitrary such profile might generate a sequence of distinct Pareto-efficient allocations that are implicated in an intensity-dominance cycle. To this end, let D be the intensity-dominance relation that is introduced in Definition 1. Suppose to the contrary that

$$w^1 D w^2 D \dots D w^k D w^1 \tag{3}$$

for Pareto efficient allocations $w^1, \dots, w^k$ on $X := \{a, b, c\}$.

Observation 1
The $n = 3$ postulate implies that for any two allocations w^i, w^{i+1} such that $w^i D w^{i+1}$ it must be that $w^i_l = w^{i+1}_l$ for exactly one agent $l \in \{1, 2, 3\}$ and $(w^i_j, w^i_k) = (w^{i+1}_k, w^{i+1}_j)$ for $j, k \neq l$.

Observation 2
The $n = 3$ postulate implies $k \leq 6$.

Observation 3
Pareto efficiency of $w^i = (a', b', c')$ and Strictness together imply

$$s_2(a', b') > 0 \implies s_1(a', b') > 0, \tag{4}$$
$$s_3(b', c') > 0 \implies s_2(b', c') > 0, \tag{5}$$
$$s_3(a', c') > 0 \implies s_1(a', c') > 0. \tag{6}$$

Observation 4
Strictness and the canonicality assumption for intensify function profile s imply $s_i(a', b') = s_i(c', d') > 0 \Leftrightarrow (a', b') = (c', d')$ and $s_i(a', b') > 0 \Leftrightarrow s_i(a', b') \in \{1, 2, 3\}$.

Notice that (3) is impossible for $k = 2$ because D is asymmetric by construction. Suppose $k = 3$. Without loss of generality, write $w^1 := (a, b, c)$ and $w^2 := (b, a, c)$. Then, In view of

Observation 1, either $w^3 = (b, c, a)$ or $w^3 = (c, a, b)$ must be true. Since, in both cases, w^1 and w^3 are D-incomparable by construction, the w^3Dw^1 postulate in (3) is contradicted.

Now suppose $k = 4$. By (3) and the above implications, we may take w^1, w^2, w^3 to be as in the $k = 3$ case, from which it then follows that allocation w^4 must satisfy either $w^4 = (c, b, a)$ or $w^4 = (a, c, b)$. Notice that each of these possibilities is compatible with $w^3 = (b, c, a)$ and with $w^3 = (c, a, b)$. We therefore have the following 4 cases to consider:

[*Remark:* in what follows we make repeated use, often without explicit reference, of the Pareto efficiency implications (4)–(6), the skew-symmetry property (1b) and, whenever exact values of the s_i functions are asserted, of the lateral-consistency property (1c) together with the assumption that every s_i is canonical and strict (cf Observations 3–4).]

Case 1. $w^3 = (b, c, a)$, $w^4 = (c, b, a)$.
By definition,

$$\begin{aligned} w^1Dw^2 &\implies s_1(a, b) > s_2(a, b), \\ w^2Dw^3 &\implies s_2(a, c) > s_3(a, c), \\ w^3Dw^4 &\implies s_1(b, c) > s_2(b, c), \\ w^4Dw^1 &\implies s_3(a, c) > s_1(a, c). \end{aligned}$$

Therefore,

$$s_2(a, c) > s_3(a, c) > s_1(a, c). \tag{7}$$

By assumption, w^i is Pareto efficient for $i \leq 4$. So, it follows from (4)–(6) that there are 4 subcases to consider:
Subcase 1α. $s_1(a, b) > s_2(a, b) > 0$ and $s_1(b, c) > s_2(b, c) > 0$. By (1c) and the fact that s_1, s_2 are canonical, this implies $s_1(a, c) = 3$, which contradicts (7).
Subcase 1β. $s_1(a, b) > s_2(a, b) > 0$ and $s_2(c, b) > s_1(c, b) > 0$. Suppose $s_1(a, c) > 0$ is also true. Then, by (1c), (7) and the fact that s_1 is canonical, $s_1(a, c) = 1$ and $s_1(a, b) = 3$. If $s_2(a, c) > 0$ is also true, then $s_2(a, b) = 3$. This contradicts $s_1(a, b) > s_2(a, b)$. So, it must be that $s_2(c, a) > 0$ instead. But in this case $s_2(c, a) > 0$, $s_2(a, b) > 0$ and (1c) together imply $s_2(c, b) = 3$. This contradicts $s_2(a, c) = 3$ which is now implied by (7) and the fact that the profile s is canonical. Thus, it must be that $s_1(c, a) > 0$ instead. So now we have $s_1(c, a) > 0$, $s_1(a, b) > 0$, which implies $s_1(c, b) = 3$. But since, by assumption, $s_2(c, b) > s_1(c, b)$ and s_2 is canonical, this is a contradiction.
Subcase 1γ. $s_2(b, a) > s_1(b, a) > 0$ and $s_1(b, c) > s_2(b, c) > 0$. Suppose first that $s_2(a, c) > 0$ is also true. Then, $s_2(b, a) > 0$ and $s_2(a, c) > 0$ implies $s_2(b, c) = 3$. If $s_1(a, c) > 0$ is also true, then (7) and the fact that s is canonical together imply $s_2(a, c) = s_2(b, c)$, which contradicts (1c) and Strictness. So, it must be that $s_1(c, a) > 0$. From $s_1(b, c) > 0$ and $s_1(c, a) > 0$ we now get $s_1(b, a) = 3$. In view of s being canonical, this contradicts $s_2(b, a) > s_1(b, a)$.
Subcase 1δ. $s_2(b, a) > s_1(b, a) > 0$ and $s_2(c, b) > s_1(c, b) > 0$. Because s is canonical, this and (1c) readily imply $s_1(c, a) = s_2(c, a) = 3$. But since (7) is equivalent to $s_1(c, a) > s_3(c, a) > s_2(c, a)$, this is a contradiction.
Hence, w^4Dw^1 is impossible for such w^3 and w^4.
Case 2. $w^3 = (c, a, b)$, $w^4 = (c, b, a)$.

By definition,

$$\begin{aligned} w^1 D w^2 &\implies s_1(a,b) > s_2(a,b), && (8)\\ w^2 D w^3 &\implies s_1(b,c) > s_3(b,c), && (9)\\ w^3 D w^4 &\implies s_2(a,b) > s_3(a,b), && (10)\\ w^4 D w^1 &\implies s_3(a,c) > s_1(a,c). && (11) \end{aligned}$$

Therefore,

$$s_1(a,b) > s_2(a,b) > s_3(a,b). \quad (12)$$

In view of (4)–(6), we can now consider the following 4 exhaustive and mutually exclusive subcases:

Subcase 2α. $s_1(a,b) > s_2(a,b) > 0$ and $s_1(b,c) > s_3(b,c) > 0$. By (12) and (4)–(6), the former postulate implies $s_3(a,b) > 0$. Since s is canonical, this further implies $s_3(a,b) = 1$, $s_2(a,b) = 2$ and $s_1(a,b) = 3$. This, canonicality of s and $s_1(b,c) > s_3(b,c) > 0$ together imply $s_1(b,c) = 2$ and $s_3(b,c) = 1 = s_3(a,b)$, which contradicts Strictness.

Subcase 2β. $s_1(a,b) > s_2(a,b) > 0$ and $s_3(c,b) > s_1(c,b) > 0$. For the same reasons as in 2α, we have $s_3(a,b) = 1$, $s_2(a,b) = 2$ and $s_1(a,b) = 3$. This, together with canonicality of s and $s_3(c,b) > s_1(c,b) > 0$, further implies $s_1(c,b) = 1$. Hence, it also follows that either $s_1(a,c) = 2$ or $s_1(c,a) = 2$. The latter possibility cannot be valid, for (1c) and $s_1(c,a) > 0$, $s_1(a,b) > 0$ would then imply $s_1(c,b) = 3$, which contradicts $s_1(c,b) = 1$. Consider then the case of $s_1(a,c) = 2$. This, together with (11) and canonicality of s, implies $s_3(a,c) = 3$. Thus, we have $s_3(a,c) = 3$, $s_3(a,b) = 1$ and, from $s_3(c,b) > s_1(c,b) > 0$ and canonicality, $s_3(c,b) = 2$. But, by (1c) and canonicality, $s_3(a,c) > 0$ and $s_3(c,b) > 0$ implies $s_3(a,b) = 3$, a contradiction.

Subcase 2γ. $s_2(b,a) > s_1(b,a) > 0$ and $s_1(b,c) > s_3(b,c) > 0$. The former postulate, together with (12) and canonicality, implies $s_3(b,a) = 3$, $s_2(b,a) = 2$, $s_1(b,a) = 1$. By (10), either $s_3(a,c) > s_1(a,c) > 0$ or $s_1(c,a) > s_3(c,a) > 0$ also holds. Consider the first possibility. From $s_1(b,a) = 1$, $s_1(a,c) > 0$, (1c) and canonicality we get $s_1(a,c) = 2$. This and (11) implies $s_3(a,c) = 3$. Since $s_3(b,a) = 3$ is also true, this contradicts Strictness. Hence, it must be that $s_1(c,a) > s_3(c,a) > 0$. But in this case $s_1(b,c) > 0$, $s_1(c,a) > 0$, (1c) and canonicality imply $s_1(b,a) = 3$, which contradicts (12).

Subcase 2δ. $s_2(b,a) > s_1(b,a) > 0$ and $s_3(c,b) > s_1(c,b) > 0$. As in 2γ, we have $s_3(b,a) = 3$, $s_2(b,a) = 2$, $s_1(b,a) = 1$. But $s_3(c,b) > 0$, $s_3(b,a) > 0$ and (1c) imply $s_3(c,a) > s_3(b,a) = 3$ which, by canonicality, is impossible.

Hence, $w^4 D w^1$ is impossible for such w^3 and w^4 too.

Case 3. $w^3 = (b,c,a)$, $w^4 = (a,c,b)$.

By definition:

$$\begin{aligned} w^1 D w^2 &\implies s_1(a,b) > s_2(a,b); && (13)\\ w^2 D w^3 &\implies s_1(b,c) > s_3(b,c); && (14)\\ w^3 D w^4 &\implies s_3(a,b) > s_1(a,b); && (15)\\ w^4 D w^1 &\implies s_3(b,c) > s_2(b,c). && (16) \end{aligned}$$

It follows that

$$s_3(a,b) > s_1(a,b) > s_2(a,b). \quad (17)$$

We proceed by considering the following 4 mutually exclusive and exhaustive subcases:
Subcase 3α. $s_1(a,b) > s_2(a,b) > 0$ and $s_1(b,c) > s_3(b,c) > 0$. The latter, together with (4)–(6), (16) and canonicality, implies $s_3(a,b) = 1$. But canonicality, (4)–(6) and (6) also implies $s_3(a,b) = 3$, which contradicts Strictness.
Subcase 3β. $s_1(a,b) > s_2(a,b) > 0$ and $s_3(c,b) > s_1(c,b) > 0$. The first postulate and (17), together with canonicality, implies $s_3(a,b) = 3$, $s_1(a,b) = 2$ and $s_2(a,b) = 1$. Since $s_3(c,b) > s_1(c,b) > 0$ is also assumed, this and canonicality further imply $s_3(c,b) = 2$. Now, because $s_3(a,b) > 0$ and $s_3(c,b) > 0$, it follows from (1c) that $s_3(c,a) > 0$ too. But (1c) in this case further implies $s_3(c,a) > s_3(a,b) = 3$, which is impossible.
Subcase 3γ. $s_2(b,a) > s_1(b,a) > 0$ and $s_1(b,c) > s_3(b,c) > 0$. The first postulate, together with (17) and canonicality, implies $s_2(b,a) = 3$, $s_1(b,a) = 2$, $s_3(b,a) = 1$. The second postulate and $s_1(b,a) = 2$, together with Strictness, implies $s_1(b,c) = 3$. This in turn implies $s_1(a,c) = 1$ or $s_1(c,a) = 1$. If the latter is true, then $s_1(b,c) > 0$, $s_1(c,a) > 0$ and (1c), together with canonicality, implies $s_1(b,a) = 3$, a contradiction. Hence, it must be that $s_1(a,c) = 1$. We therefore have $s_1(b,a) = 2$, $s_1(a,c) = 1$ and, by (1c) and canonicality, $s_1(b,c) = 3$. From (14), (16), (4)–(6) and canonicality we also know that $s_1(b,c) > s_3(b,c) > s_2(b,c) > 0$ implies $s_3(b,c) = 2$ and $s_2(b,c) = 1$. Thus, we have $s_3(b,a) > 0$, $s_3(b,c) > 0$ and, by (4)–(6) and $s_1(a,c) > 0$, also $s_3(a,c) > 0$. But $s_3(b,a) > 0$, $s_3(a,c) > 0$ together with (1c) and canonicality implies $s_3(b,c) = 3$, a contradiction.
Subcase 3δ. $s_2(b,a) > s_1(b,a) > 0$ and $s_3(c,b) > s_1(c,b) > 0$. These readily imply $s_1(c,a) = 3$. As above, (17) implies $s_2(b,a) = 3$, $s_1(b,a) = 2$ and $s_3(b,a) = 1$. By (1c), $s_1(c,a) = 3$ and $s_1(a,b) = 2$ implies $s_1(c,b) = 1$. From the above postulates and from (16), $s_2(c,b) > s_3(c,b) > s_1(c,b)$ further implies $s_2(c,b) = 3$, which contradicts $s_2(b,a) = 3$ and Strictness.
Hence, $w^4 D w^1$ is impossible for such w^3 and w^4 here as well.
Case 4. $w^3 = (c,a,b)$, $w^4 = (a,c,b)$.
By definition:

$$w^1 D w^2 \implies s_1(a,b) > s_2(a,b); \tag{18}$$
$$w^2 D w^3 \implies s_1(b,c) > s_3(b,c); \tag{19}$$
$$w^3 D w^4 \implies s_2(a,c) > s_1(a,c); \tag{20}$$
$$w^4 D w^1 \implies s_3(b,c) > s_2(b,c). \tag{21}$$

Observe now that

$$s_1(b,c) > s_3(b,c) > s_2(b,c). \tag{22}$$

Suppose first that $s_2(b,c) > 0$. Then, by (22) and (4)–(6), $s_1(b,c) = 3$, $s_2(b,c) = 1$ and $s_3(b,c) = 2$. From $s_1(b,c) = 3$ and (1c) we also get $s_1(b,a) > 0$ and $s_1(a,c) > 0$. Hence, by (18), $s_2(b,a) > s_1(b,a) > 0$ and, by (20), $s_2(a,c) > s_1(a,c) > 0$. These inequalities and (1c) together imply $s_2(b,c) = 3$, which is a contradiction.
Now suppose instead that $s_2(b,c) < 0$, i.e. $s_2(c,b) > 0$. It follows from (22) that $s_2(c,b) = 3$, $s_3(c,b) = 2$ and $s_1(c,b) = 1$. Suppose $s_1(a,c) > 0$. From (20) and (4)–(6), $s_2(a,c) > 0$. Since $s_2(a,c) > 0$ and $s_3(c,b) > 0$, by (1c) we get $s_2(a,b) > s_2(c,b) = 3$, which is impossible. Hence, $s_1(c,a) > 0$ holds instead and, from (20) and (4)–(6), $s_1(c,a) > s_2(c,a) > 0$ is also true. Suppose $s_2(a,b) > 0$ holds too. By (18), $s_1(a,b) > 0$. By (1c) and $s_1(c,a) > 0$, $s_1(a,b) > 0$ we get $s_1(c,b) = 3$, a contradiction. Hence, $s_2(b,a) > 0$ must be true instead and, by (18), $s_2(b,a) > s_1(b,a) > 0$ also. So, we have $s_2(c,b) > 0$ and $s_2(b,a) > 0$, which, by (1c), implies $s_2(c,a) > s_(c,b) = 3$. This too is a contradiction.

Hence, $w^4 D w^1$ is impossible for such w^3 and w^4 also.

Next, suppose $k = 5$. Arguing as above, allocations $w^1, \ldots, w^4$ in (3) must be as in one of the four cases considered previously. Combined with the fact that each w^i in sequence $(w^1, \ldots, w^5)$ must be distinct and the notational convention $w^1 = (a, b, c)$ and $w^2 = (b, a, c)$, this gives rise to the following possibilities:

$$\begin{array}{lll} w^3 = (b,c,a), & w^4 = (c,b,a), & w^5 = (c,a,b); \\ w^3 = (c,a,b), & w^4 = (c,b,a), & w^5 = (b,c,a); \\ w^3 = (b,c,a), & w^4 = (a,c,b), & w^5 = (c,a,b); \\ w^3 = (c,a,b), & w^4 = (a,c,b), & w^5 = (b,c,a). \end{array}$$

Clearly, because either $w^5 = (b, c, a)$ or $w^5 = (c, a, b)$ must be true in all four cases, and recalling that $w^1 = (a, b, c)$ by assumption, $w^5 D w^1$ cannot happen.

Finally, suppose $k = 6$. With allocations $w^1, \ldots, w^5$ in (3) being as in the $k = 5$ case that was just considered above, w^6 can only coincide with allocation (a, c, b) in each of the four relevant cases. In view of the previous steps, these are as follows:

Case 1: $w^1 = (a, b, c)$, $w^2 = (b, a, c)$, $w^3 = (b, c, a)$, $w^4 = (c, b, a)$, $w^5 = (c, a, b)$, $w^6 = (a, c, b)$. By definition:

$$\begin{array}{lll} w^1 D w^2 & \Longrightarrow & s_1(a,b) > s_2(a,b), \\ w^2 D w^3 & \Longrightarrow & s_2(a,c) > s_3(a,c), \\ w^3 D w^4 & \Longrightarrow & s_1(b,c) > s_2(b,c), \\ w^4 D w^5 & \Longrightarrow & s_3(a,b) > s_2(a,b), \\ w^5 D w^6 & \Longrightarrow & s_2(a,c) > s_1(a,c), \\ w^6 D w^1 & \Longrightarrow & s_3(b,c) > s_2(b,c). \end{array}$$

It follows from the above that

$$s_2(a,c) > s_1(a,c) > s_1(a,b) > s_2(a,b). \tag{23}$$

Suppose $s_2(a, b) > 0$. Then, (23) implies $s_2(a, c) > 3$, which contradicts canonicality. If $s_2(b, a) > 0$ instead, then (23) together with (1b) implies $s_2(b, a) > 3$ and results in the same contradiction.

Case 2: $w^1 = (a, b, c)$, $w^2 = (b, a, c)$, $w^3 = (c, a, b)$, $w^4 = (c, b, a)$, $w^5 = (b, c, a)$, $w^6 = (a, c, b)$. Notice that the following postulated dominance implications

$$\begin{array}{lll} w^1 D w^2 & \Longrightarrow & s_1(a,b) > s_2(a,b), \\ w^3 D w^4 & \Longrightarrow & s_2(a,b) > s_3(a,b), \\ w^5 D w^6 & \Longrightarrow & s_3(a,b) > s_1(a,b) \end{array}$$

lead to $s_1(a, b) > s_2(a, b) > s_3(a, b) > s_1(a, b)$, which is absurd.

Case 3: $w^1 = (a, b, c)$, $w^2 = (b, a, c)$, $w^3 = (b, c, a)$, $w^4 = (a, c, b)$, $w^5 = (c, a, b)$, $w^6 = (a, c, b)$. Observe here that the postulated dominance implications

$$\begin{array}{lll} w^4 D w^5 & \Longrightarrow & s_1(a,c) > s_2(a,c), \\ w^5 D w^6 & \Longrightarrow & s_2(a,c) > s_1(a,c) \end{array}$$

directly contradict each other.

Case 4: $w^1 = (a, b, c)$, $w^2 = (b, a, c)$, $w^3 = (c, a, b)$, $w^4 = (a, c, b)$, $w^5 = (b, c, a)$, $w^6 = (a, c, b)$.

Notice that, as in Case 2, the postulated dominance implications

$$\begin{aligned} w^4 D w^5 &\implies s_1(a,b) > s_3(a,b), \\ w^5 D w^6 &\implies s_3(a,b) > s_1(a,b) \end{aligned}$$

result in the same contradiction.

Since we have shown that D is acyclic when $n = 3$, this and the fact that allocations are finitely many together imply that an intensity-efficient allocation always exists in this case. ■

We illustrate intensity-dominance and intensity-efficiency with an example where $n = 4$.

Example. Let $X := \{a, b, c, d\}$ and consider the intensity profile $\dot{\succsim} = (\dot{\succsim}_1, \dot{\succsim}_2, \dot{\succsim}_3, \dot{\succsim}_4)$ that is represented canonically by

$$\begin{array}{cccc} \overbrace{\underbrace{\begin{array}{c} s_1(a,d) = 6 \\ s_1(b,d) = 5 \\ s_1(a,c) = 4 \\ s_1(b,c) = 3 \\ s_1(a,b) = 2 \\ s_1(c,d) = 1 \end{array}}_{\substack{\textit{not utility-difference} \\ \textit{decomposable}}}} & \begin{array}{c} s_2(d,a) = 6 \\ s_2(d,c) = 5 \\ s_2(d,b) = 4 \\ s_2(c,a) = 3 \\ s_2(c,b) = 2 \\ s_2(b,a) = 1 \end{array} & \begin{array}{c} s_3(a,d) = 6 \\ s_3(a,c) = 5 \\ s_3(a,b) = 4 \\ s_3(b,d) = 3 \\ s_3(c,d) = 2 \\ s_3(b,c) = 1 \end{array} & \begin{array}{c} s_4(d,a) = 6 \\ s_4(c,a) = 5 \\ s_4(d,b) = 4 \\ s_4(c,b) = 3 \\ s_4(b,a) = 2 \\ s_4(d,c) = 1 \end{array} \end{array}$$

Notice first that the induced preference profile $\succ := (\succ_1, \succ_2, \succ_3, \succ_4)$ is such that

$$\begin{array}{ccccccc} a & \succ_1 & b & \succ_1 & c & \succ_1 & d \\ d & \succ_2 & c & \succ_2 & b & \succ_2 & a \\ a & \succ_3 & b & \succ_3 & c & \succ_3 & d \\ d & \succ_4 & c & \succ_4 & b & \succ_4 & a \end{array}$$

Notice further that the set of Pareto efficient allocations corresponding to $\succ$ is

$$\Big\{ \underbrace{(a,c,b,d)}_{w}, \underbrace{(a,d,b,c)}_{x}, \underbrace{(b,c,a,d)}_{y}, \underbrace{(b,d,a,c)}_{z} \Big\}$$

Notice, finally, that zDw holds here because $s_3(a,b) > s_1(a,b)$ and $s_4(c,d) > s_2(c,d)$; zDw because $s_3(a,b) > s_1(a,b)$; and zDy because $s_4(c,d) > s_2(c,d)$.
Thus, z is the unique intensity-efficient allocation. ◊

While we conjecture that existence of intensity-efficient allocations holds generically when $n = 4$, we know that this is not so with larger numbers of agents and items.

Theorem 2
When $n \geq 5$ there are intensity profiles for which no intensity-efficient allocation exists.

Proof of Theorem 2.
Consider the example strict intensity profile on $X = \{a, b, c, d, e\}$ that is shown in Table

Table 2: An intensity profile when $n = 5$ for which no intensity-efficient allocation exists.

Agent $i =$ / $s_i(a', b') =$	1	2	3	4	5
10	(a, e)	(a, e)	(a, e)	$(d, \cdot)$	$(e, \cdot)$
9	(a, d)	(a, d)	(a, d)		
8	(b, e)	(b, e)	(a, c)		
7	(b, d)	(a, c)	(b, e)		
6	(a, c)	(b, d)	(b, d)		
5	(c, e)	(c, e)	(c, e)		
4	(a, b)	(c, d)	(c, d)		
3	(b, c)	(a, b)	(d, e)		
2	(c, d)	(b, c)	(a, b)		
1	(d, e)	(d, e)	(b, c)		

2.[6] The preferences induced by this profile are

$$a \succ_i b \succ_i c \succ_i d \succ_i e$$

for $i = 1, 2, 3$, but the intensity orderings of these agents are distinct. On the other hand, the last two columns are meant to be read as suggesting that agents 4 and 5 prefer d and c over everything else, respectively. Other than that, the preferences and intensities of these two agents are inconsequential for the argument.

In view of the above, the Pareto-efficient allocations that correspond to the preferences induced by $(s_1, \ldots, s_5)$ are

$$\underbrace{(c, a, b, d, e)}_{s}, \quad \underbrace{(c, b, a, d, e)}_{t}, \quad \underbrace{(a, b, c, d, e)}_{x}, \quad \underbrace{(a, c, b, d, e)}_{y}, \quad \underbrace{(b, c, a, d, e)}_{w}, \quad \underbrace{(b, a, c, d, e)}_{z}.$$

Observe now that:

$$\overbrace{(c, a, b, d, e) D (c, b, a, d, e)}^{s_2(a,b) > s_3(a,b)},$$

$$\overbrace{(c, b, a, d, e) D (a, b, c, d, e)}^{s_3(a,c) > s_1(a,c)},$$

$$\overbrace{(a, b, c, d, e) D (a, c, b, d, e)}^{s_2(b,c) > s_3(b,c)},$$

$$\overbrace{(a, c, b, d, e) D (b, c, a, d, e)}^{s_1(a,b) > s_3(a,b)},$$

$$\overbrace{(b, c, a, d, e) D (b, a, c, d, e)}^{s_2(a,c) > s_2(a,c)},$$

$$\overbrace{(b, a, c, d, e) D (c, a, b, d, e)}^{s_1(b,c) > s_3(b,c)}.$$

Hence, the cycle $sDtDxDyDwDzDs$ implies that there is no intensity-efficient allocation.■

[6]

This somewhat surprising fact invites an informal analogy to be drawn between intensity-dominance cycles over *allocations* with at least 5 agents and Condorcet cycles over *alternatives* in majority-based preference aggregation with at least 3 agents (Sen, 2017/1970). Unlike that framework, however, although intensity-dominance cycles here may prevent refining the Pareto set, they do not lead to a "policy paralysis" problem because Pareto allocations always exist and, absent any distributionally juster suggestions, one of them might be promoted by the social planner.

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